Preparation for EMC 2024

Fourth Training Test for Junior Category

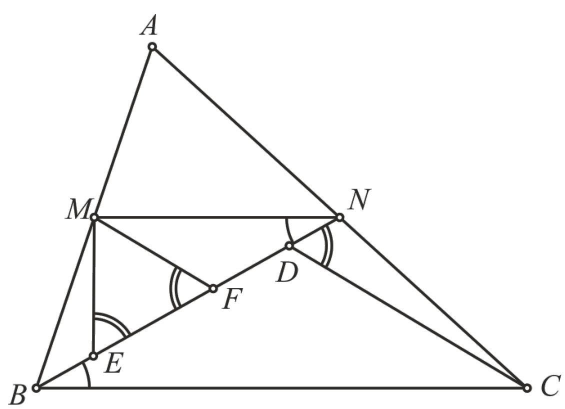
Solutions

Problem 1. In acute-angled $\triangle ABC$ with $AB < BC$, points M, N are the midpoints of sides AB, AC respectively. Let D and E be two points on the segment BN such that $CD = 2ME$ and $BE < ED$. Prove that $\angle NEM = \angle CDN$.

Solution. Let F on BN be such that $BD = 2NE$. Since $BE < ED$, $BC = 2NM$, and $\angle CBD = \angle MNF$, we have hence $\triangle BCD \sim \triangle NMF$, hence $MF = \frac{1}{2}CD = ME$ and

$$\angle CDN = 180^\circ - \angle BDC = 180^\circ - \angle NFM = \angle MFB.$$

Since $\triangle MEF$ is isosceles we conclude $\angle CDN = \angle MFB = \angle NEM$. $\square$



Problem 2. Find all positive integers n for which

$$\frac{2^{n!}-1}{2^n-1}$$

is a perfect square.

Solution. For $n=1$ and $n=2$ we have

$$\frac{2^{1!}-1}{2^1-1}=\frac{2^{2!}-1}{2^2-1}=1,$$

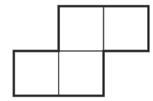
which is a perfect square. Similarly for $n=3$ we have

$$\frac{2^{3!}-1}{2^3-1}=\frac{2^6-1}{2^3-1}=2^3+1=9=3^2.$$

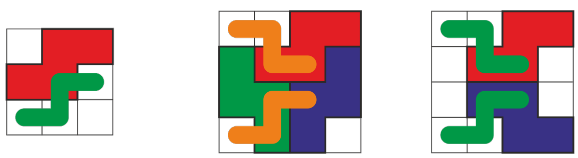
For $n\geq 4$ we have $2\mid\frac{n!}{2}$ and $n\mid\frac{n!}{2}$, hence $2^{\frac{n!}{2}}$ is a perfect square and $2^n-1\mid 2^{\frac{n!}{2}}-1$. On the other hand $2^{\frac{n!}{2}}-1$, and $2^{\frac{n!}{2}}+1$ are odd, hence coprime numbers. This means that $2^{\frac{n!}{2}}+1$, (as well as $2^{\frac{n!}{2}}$) must be perfect squares, which is impossible.

We conclude that the claim is true only for $n\in\{1,2,3\}$. $\square$

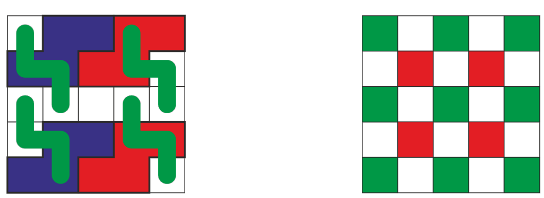
Problem 3. Let $n \geq 3$ be an integer. In a square of size $n \times n$ we place the shapes on the picture on the right, such that the unit squares coincide. On each unit square it is allowed to overlap at most two figures, rotations and flips are allowed, and no figure can exit the borders of the square. For each n find the least number of unit squares that must be left uncovered (a square is covered if at least one figure covers it).



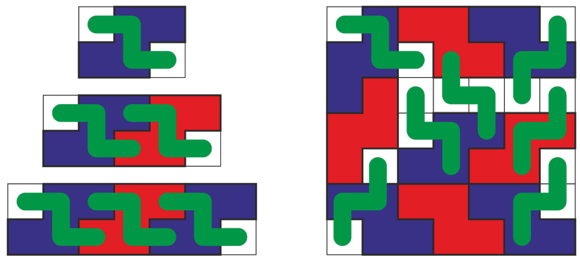
Solution. For $n = 3$ every figure must cover the middle square, hence we can place at most two figures covering at most $2 \cdot 4 - 1 = 7$ unit squares shown on the left picture. This gives us the minimum of 2 uncovered unit squares. Similarly for $n = 4$ the minimum is 1 shown on the middle picture. If we suppose that all unit squares can be covered, than we have at least one figure covering each corner. However these figures cover two of the middle squares and one more edge square, hence covering 12 squares in total. After this we cannot add a figure that doesn't cower a middle square for the third time, hence leaving 4 uncovered unit squares (an example can be seen on the right picture).



For $n = 5$ again we have a solution with only one uncovered unit square (on the left). If we suppose that we can cover all unit squares again we have four figures for each corner. In the right picture with red we have 4 unit squares such that each must cover one of them, and in green we have 9 unit squares such that no two can be covered by the same figure. The red once imply that at most 8 figures can be put in the square, and the green ones that at least 9 figures are needed to cover all squares. Hence, at least one unit square must be left uncovered.



On the next picture we give complete coverings of 2×3, 2×5, 2×7, and 7×7 boards. Since every square with side $n \geq 6$ can be split into such boards we can completely cover a square with side $n \geq 6$.



We conclude that for $n = 3$ the minimum number of uncovered squares is 2, for $n = 4$ and $n = 5$ it is 1, and for every $n \geq 6$ this number is 0. $\square$

Problem 4. Let $a \geq b \geq c \geq d$ be positive real numbers. Prove that

$$\frac{b^3}{a} + \frac{c^3}{b} + \frac{d^3}{c} + \frac{a^3}{d} + 3\left(ab + bc + cd + da\right) \geq 4\big(a^2 + b^2 + c^2 + d^2\big).$$

When does the equality hold?

Solution. Since $\frac{(b-a)^3}{a} = \frac{b^3}{a} - 3b^2 + 3ab - a^2$, the given inequality is equivalent to

$$\frac{(b-a)^3}{a} + \frac{(c-b)^3}{b} + \frac{(d-c)^3}{c} + \frac{(a-d)^3}{d} \geq 0.$$

However by the given conditions $a - d \geq 0$ and $b - a, c - b, d - c \leq 0$, hence it is enough to prove that $(a-d)^3 \geq (a-b)^3 + (b-c)^3 + (c-d)^3$. This is true since $a - d = (a-b) + (b-c) + (c-d)$.
The equality can hold only when $a = b = c = d$. □