Preparation for EMC 2024

Fourth Training Test for Senior Category

Solutions

Problem 1. On an infinite chessboard consisting of unit squares (x, y) with $x, y \geq 0$ two players play the following game: initially a king is positioned somewhere on the board, but not on $(0, 0)$, and they alternatively move it either down or left or down-left. The player Who loses is the one who moves the king into the $(0, 0)$ square. Find the initial position of the king for which the first player wins.

Solution. We will prove by induction on $x + y \geq 1$ that the first player has a winning strategy if and only if either: (1) $x = 0$ and y is even, or (2) $y = 0$ and x is even, or (3) $x, y \geq 1$ and not both x and y are even.

Clearly for $x + y = 1$, the first player is forced to move into $(0, 0)$ and lose. Since the pairs $(0, 1)$ and $(1, 0)$ are not of the form above, this proves the base case.

For $x + y > 1$, if one of x and y is 0, then without loss of generality let $x = 0$. The first player has to move into $(0, y - 1)$ so, by induction, he wins if and only if y is even.

If $x = 1$ or $y = 1$, then without loss of generality let $x = 1$, so the first player can move into one of $(0, y)$ and $(0, y - 1)$ which gives him a winning strategy.

Suppose $x, y \geq 2$. If x and y are both even, the first player has to move into one of $(x - 1, y)$, $(x - 1, y - 1)$ or $(x, y - 1)$, all of which have positive coordinates that are not both even, so, by induction, he loses. Else, one of $(x - 1, y)$, $(x - 1, y - 1)$ and $(x, y - 1)$ has even, non-negative coordinates, so he can move into that one, winning by induction. $\square$

Problem 2. Let n be a positive integer. What is the largest number of elements that one can choose from the set $A = \{1, 2, \dots, 2n\}$ such that the sum of any two chosen numbers is composite?

Solution. We prove by induction on n that if we choose $n+1$ numbers from A, then there exist two numbers whose sum is prime.

When $n = 1$, there is nothing to prove, as $1 + 2 = 3$ is prime.

Assume now that the statement holds for all integers $1 \le n \le m - 1$, $m \ge 2$, and we show that it also holds for $n = m$.

By Chebyshev's theorem, there is a prime number in the interval $(2m, 4m)$, call it p. Let $p = 2m+k$, for some $k > 0$. Then k is odd and $k-1 \le 2(m-1)$. Notice that if we choose $m + 1$ numbers from the set $\{1, 2, \dots, 2m\}$, we either choose more than half of elements in the set $\{1, 2, \dots, k - 1\}$ or by Pigeonhole Principle, we choose both elements of one of the following pairs:

$$\{k, 2m\}, \{k, 2m - 1\}, \dots, \{\frac{k + 2m - 1}{2}, \frac{k + 2m + 1}{2}\}.$$

In the first case, the conclusion follows from the induction hypothesis applied to $n = (k - 1)/2$, while in the second case the result holds because the sum of elements in each of the above pairs is p. This completes our induction.

To finish the question, notice that an example with n elements is given by choosing the elements $2, 4, 6, \dots, 2n$. $\square$

Problem 3. Prove that for every positive integer n there exists an n-digit number divisible by 5^n, all of whose digits are odd.

Solution. Arguing by contradiction, suppose n is the smallest counter-example. Clearly, $n \geq 2$. Let A be a number comprised from $n-1$ odd digits, which is divisible by 5^{n-1}. Notice that each of the numbers $1, 3, 5, 7, 9$ gives a different residue modulo 5. Since there are just five residues modulo 5, they must cover all residues modulo 5. Hence there exists $c \in \{1, 3, 5, 7, 9\}$ such that

$$c \cdot 2^{n-1} \equiv -\frac{A}{5^{n-1}} \pmod{5}$$

and so

$$c \cdot 10^{n-1} \equiv -A \pmod{5^n}.$$

However then the number $10^{n-1}c + A$ is an n-digit number divisible by 5^n all of whose digits are odd, a contradiction. $\square$

Problem 4. Determine all functions $f : \{1, 2, 3, \ldots\} \to \{1, 2, 3, \ldots\}$ which satisfy the property that for all positive integers a and b, there exists a nondegenerate triangle with sides of lengths

$$a, f(b) \text{ and } f(b + f(a) - 1)\,.$$

(A triangle is non-degenerate if its vertices are not collinear.)

Solution. First notice that if a triangle has sides of lengths $1, a, b$ with a, b positive integers, then by the triangle inequality we must have that $a = b$. For $a = 1$, the above remark tells us that $f(b) = f(b + f(l) - 1)$.

Notice that $f(1) = 1$, as otherwise $f(1) - 1 > 0$, which from the above relation means that f repeats itself every $f(1) - 1$ numbers. This means that f can take only finitely many values, so if we take a sufficiently large, $a, f(b), f(b+f(a)|1)$ cannot be a triangle, because $a - f(b) > f(b+f(a)-1)$.

Now, setting $b = 1$, we obtain that $a, 1, f(f(a))$ must be the sides of a triangle. This implies that $f(f((a)) = a$ by one of the previous remarks.

Claim. $f(n) = (n-1)f(2) - (n-2)$, *for every* $n \geq 3$.

Proof. From $f(f(a)) = a$, f is bijective, so we now know that $a, b, f(f(a) + f(b) - 1)$ can be the side lengths of a triangle. This implies that

$$f(f(a) + f(b) - 1) < a + b\,.$$

If we take $a = b = 2$ we then obtain that $f(2f(2)-1) < 4$, i.e. $f(2f(2) - 1) \in \{1, 2, 3\}$. The value 1 is not possible, as we would have $2f(2) - 1 = 1$, i.e. $f(2) = 1$, contradicting the bijectivity of f. The value 2 is also not possible, since we would get $2f(2) - 1 = f(2)$, i.e. $f(2) = 1$, which is again a contradiction. Therefore, we must have $2f(2) - 1 = f(3)$, proving the base case $n = 3$.

For the induction step we use a similar argument, by taking $a = 2, b = n$, and arguing that $f(f(2) + f(n)|1) = n + 1$. $\diamond$

We have thus the result $f(n) = (n-1)f(2) - (n-2)$. In particular, this tells us that f is strictly increasing. Since we already saw that f is bijective and $f(1) = 1$, this means we must have $f(2) = 2$. Hence the claim reads $f(n) = 2(|1) - (n-2) = n$ for all $n \geq 3$. This completes our proof, showing that $f(n) = n$ is the only solution. $\square$