Preparation for EMC 2024

Third Training Test for Junior Category

Solutions

Problem 1. Let a, b, and c be real numbers greater or equal to 1. Prove that

$$\frac{ab}{c-1}+\frac{bc}{a-1}+\frac{ca}{b-1}\geq 12\,.$$

When does equality hold?

Solution. By AM-GM we have $\frac{(c-1)+1}{2}\geq\sqrt{(c-1)\cdot 1}$, hence $\frac{c^2}{4}\geq c-1$ and similarly $\frac{a^2}{4}\geq a-1$, and $\frac{b^2}{4}\geq b-1$. Applying these on the left side we get

$$\frac{ab}{c-1}+\frac{bc}{a-1}+\frac{ca}{b-1}\geq\frac{4ab}{c^2}+\frac{4bc}{a^2}+\frac{4ca}{b^2}\geq 3\sqrt[3]{\frac{4ab}{c^2}\frac{4bc}{a^2}\frac{4ca}{b^2}}=12,$$

where we again use AM-GM for three positive terms.

By the first AM-GM the equality will hold only when $a=b=c=2$. $\square$

Problem 2. In a country there are only four type of coins valued at 74, 87, 111, and 124 dollars respectively. In how many different ways can one pay exactly 2023 dollars?
Two ways are different if there are different numbers of coins of at least one type between them.

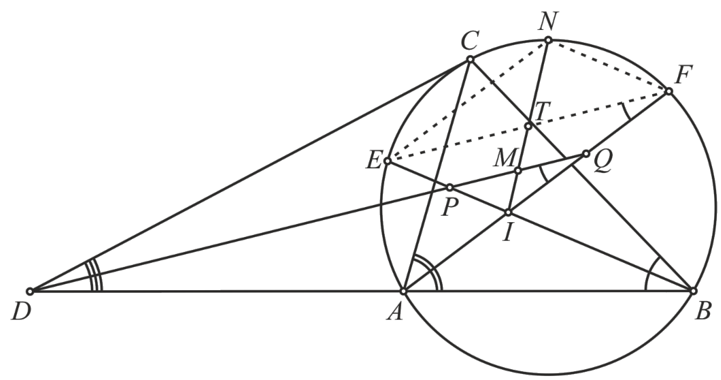
Solution. Let a, b, c, and d be the numbers of coins of each type. Since $74 \equiv 111 \equiv 0 \pmod{37}$ and $87 \equiv 124 \equiv 13 \pmod{37}$, we have $13(b+d) \equiv 2023 \equiv 25 \pmod{37}$. Hence $b+d \equiv 19 \pmod{37}$. If $b+d > 19$, then $b+d \geq 56$ and the value is at least $56 \cdot 87 > 2023$. This means that $b+d = 19$, with value of $19 \cdot 87 + 37d = 247 + 37 \cdot (38+d)$. Hence the coins of the first and third type have total value $2023 - 247 - 37 \cdot (38+d) = 37 \cdot (10-d)$. Now we have no solution for $d = 9$, one solution for each $d \in \{10, 8, 7, 6, 5, 3\}$ and two solutions for each $d \in \{4, 2, 1, 0\}$, for a total of 14 solutions. $\square$
Comment. The solutions are given with the quadruplets: $(0,9,0,10)$, $(1,11,0,8)$, $(0,12,1,7)$, $(2,13,0,6)$, $(1,14,1,5)$, $(3,15,0,4)$, $(0,15,2,4)$, $(2,16,1,3)$, $(4,17,0,2)$, $(1,17,2,2)$, $(3,18,1,1)$, $(0,18,3,1)$, $(5,19,0,0)$, and $(2,19,2,0)$.

Problem 3. Let $\triangle ABC$ be scalene with circumcircle ω and incentre I. The tanget of ω at C meets AB at D and the bisector of $\angle BDC$ meets BI and AI at P and Q respectively. If M is the midpoint of the segment PQ, prove that the line IM passes through the midpoint of the arc AB of ω that contains C.

Solution. Without loss of generality we can assume $AC < BC$, hence A is between B and D. Let E and F be the second intersection points of BI and AI with ω. Since AI and BI are the bisectors, E and F are the midpoints of the smaller ω arcs BC and CA, respectively. For the angles we have

$$2\angle DQA = 2\angle BAQ - 2\angle BDQ = \angle BAC - \angle ADC = \\ \angle DCA = \angle CAB = 2\angle ABF = 2\angle AEF,$$

hence $\angle AQD = \angle AEF$ and $EF \parallel QP$.



Let T be the midpoint of FE and N be the midpoint of the arc AB of ω that contains C. Since M is the midpoint of PQ and $PQ \parallel FE$, I, M, and T are collinear. Since

$$\angle NEA = \angle NBA = \frac{1}{2}(\angle ACB + \angle BAC) = \angle IBA + \angle BAI = \angle FIA,$$

lines EN and IF are parallel. Similarly $FN \parallel IE$, hence $IENF$ is parallelogram. This means that I, T, and N are collinear, hence IM passes through N. Since MC is tangent to circle k at C, then by the power of a point theorem we have $\square$

Problem 4. Determine all triplets of integers (p, x, y), such that p is a prime, $p - 1 = x^2$, and $2p^2 - 1 = y^2$.

Solution. For $p = 2$ we get $y^2 = 7$ is not a perfect square.
Similarly for $p = 3$ we get $x^2 = 2$, a contradiction.
For $p \geq 5$ we have $x^2 \equiv y^2 \equiv -1 \pmod{p}$. Hence

$$x \equiv y \pmod{p} \text{ or } x \equiv -y \pmod{p}.$$

Since $x < y < p\sqrt{2}$ in the first case we have $y = x + p$. Substituting we get

$$2p^2 - 1 = (x + p)^2 = x^2 + 2xp + p^2 = p - 1 + 2xp + p^2,$$

hence $p^2 = p(2x + 1)$. This implies that $x^2 + 1 = 2x + 1$, and we get the solution $x = 2$, $p = 5$, and $y = 7$.
Similarly for the second case we have $x + y = 2p$, hence

$$2p^2 - 1 = y^2 = (2p - x)^2 > (2p - \sqrt{p})^2 = 4p^2 - 4p\sqrt{p} + p.$$

This implies that $p - 1 > 2p(p - 2\sqrt{p} + 1) = 2p(\sqrt{p} - 1)^2 > 2p$, which is a contradiction.
We conclude that $(p, x, y) = (5, 2, 7)$ is the only triplet satisfying the problem conditions. $\square$