Preparation for EMC 2024

Third Training Test for Senior Category

Solutions

Problem 1. An infinite sequence $a_1, a_2, \ldots$ of positive integers is such that $a_n \geq 2$ and a_{n+2} divides $a_{n+1} + a_n$ for all $n \geq 1$. Prove that there exists a prime which divides infinitely many terms of the sequence.

Solution. Assume that every prime divides only finitely many terms of the sequence. In particular this means that there exists an integer $N > 1$ such that $2 \nmid a_n$ for all $n \geq N$. Let $M = \max(a_N, a_{N+1})$. We will now show by induction that $a_n \leq M$ for all $n \geq N$. This is obvious for $n = N$ and $n = N+1$. Now let $n \geq N+2$ be arbitrary and assume that $a_{n-1}, a_{n-2} \leq M$. By the definition of N, it is clear that a_{n-2}, a_{n-1}, a_n are all odd and so $a_n \neq a_{n-1} + a_{n-2}$, but we know that $a_n \mid a_{n-1} + a_{n-2}$ and therefore

$$a_n \leq \frac{a_{n-1} + a_{n-2}}{2} \leq \max(a_{n-1}, a_{n-2}) \leq M$$

by the induction hypothesis. This completes the induction. This shows that the sequence is bounded and therefore there are only finitely many primes which divide a term of the sequence. However there are infinitely many terms, that all have a prime divisor, hence some prime must divide infinitely many terms of the sequence. □

Problem 2. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x, y \in \mathbb{R}^+$,

$$f(xy + f(x)) = xf(y) + 2\,.$$

Solution. Make the following substitutions to the equation:

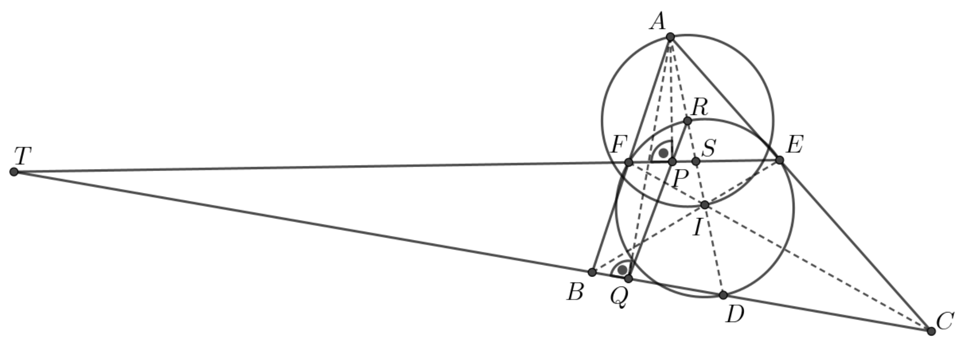
1. $(x, 1) \to f(x + f(x)) = xf(1) + 2$
2. $(1, x + f(x)) \to f(x + f(x) + f(1)) = f(x + f(x)) + 2 = xf(1) + 4$
3. $(x, 1 + \frac{f(1)}{x}) \to f(x + f(x) + f(1)) = xf\left(1 + \frac{f(1)}{x}\right) + 2$

It then follows from (2) and (3) that $f(1 + \frac{f(1)}{x}) = f(1) + \frac{2}{x}$, so we know that this function is linear for $x > 1$. Substitute $f(x) = ax + b$ and solve for a and b in the functional equation; we find that $f(x) = x + 1 \forall x > 1$.

Now, we can let $x > 1$ and $y \leq 1$. Since $f(x) = x+1$, $xy + f(x) > x > 1$, so $f(xy + f(x)) = xy + x + 2 = xf(y) + 2$. It becomes clear then that $f(y) = y + 1$ as well, so $f(x) = x + 1$ is the only solution to the functional equation. □

Problem 3. Let ABC be a triangle with $\angle BAC = 60°$; AD, BE, and CF be its bisectors; P, Q be the projections of A to EF and BC respectively; and R be the second common point of the circle DEF with AD. Prove that P, Q, R are collinear.

Solution.



It is known that the circle passing through the feet of bisectors passes also through the Feuerbach point. Also, since $\angle BAC = 60°$, the orthocenter and the circumcenter of the triangle are symmetric with respect to the bisector of $\angle BAC$. Hence the center of the nine-points circle lies on AD, i.e. the Feuerbach point coincides with R. Also, if I, r are the incenter and the inradius then $AI = 2r = 2IR$. Thus we have to prove that PQ bisects AI. Let us prove this for an arbitrary triangle.

Let EF meet AD and BC at points S, T respectively. Since the quadruple A, I, S, D is harmonic, the inversion about the circle with diameter AI swaps points S and D. On the other hand, T is the foot of the external bisector of angle A, therefore AQ and AP are the altitudes of right-angled triangles DAT and SAT. Hence $TS \cdot TP = TD \cdot TQ = TA^2$ and the inversion with center T and radius TA swaps S and P, T and Q. Since this circle and the circle with diameter AI are perpendicular, the inversion about the last circle swaps P and Q, i.e. PQ passes through its center R. □

Problem 4. There is a sheet of paper (like this one) on an infinite blackboard. Marvin secretly chooses a convex 2024− gon P that lies fully on the piece of paper. Tigerin wants to find the vertices of P. In each step, Tigerin can draw a line g on the blackboard that is fully outside the piece of paper, then Marvin replies with the line h parallel to g that is the closest to g which passes through at least one vertex of P. Prove that there exists a positive integer n such that Tigerin can always determine the vertices of P in at most n steps.

Solution 1. One of the key observations is the following. If 3 answer lines intersect at a common point X, then X must be a vertex of P.

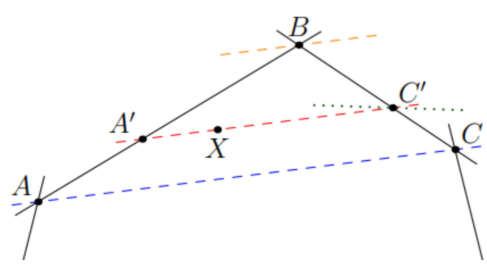
Let us start by querying the four sides of the paper. This determines a rectangle on the paper which contains P and each of the sides of the rectangle contain at least one vertex of P.

Let us assume that Q is the convex polygon given by the intersection of the closed half-planes with the answered line boundaries obtained so far containing P. Let's call a vertex of Q good if at least 3 answered lines have passed through it already, and call it bad otherwise.

If we have not found all vertices of P yet, i.e. there are less than 2024 good vertices, then there must be still at least one bad vertex of Q, since P is inside Q and if all vertices of Q were good, P couldn't have more vertices than Q.

Now suppose that we have not found all vertices of P yet. Then we repeat the following until we have not found all vertices of P yet. Pick B, a bad vertex of Q, and let A and C be its neighbours on Q, and query for a line parallel with AC outside the paper in half-plane AC containing B. We claim that repeatedly querying this way results in us finding all vertices of P in a bounded number of steps.

Let us look at the different cases based on which vertex (or possibly 2 vertices) of P the answer to this query contains.



Case 1: The answer passes through B (orange). Then B is a new good vertex. Therefore this case can happen at most 2024 times.

Case 2: The answer is the line AC itself (blue). Then A and C are points of P. We might or might not have known this before, but now we additionally know that they are direct neighbors on P. Therefore this case can happen at most 2024 times as we didn't know they were direct neighbours before this query.

Case 3: At least one vertex X of P on the answered line is in the interior of Q. In this case the answer intersects AB in A' and BC in C' (red line on the figure above). Once a vertex of P is on the boundary of Q, it can never become an interior point of Q again, therefore this case can happen at most 2024 times as X was not on the boundary of Q before this query.

Case 4: The vertex (or vertices) of P on the answered line is (are) in the line segment AB or BC, excluding endpoints. In this case the answer intersects AB in A' and BC in C' (red line on the figure above) and so either A' or C' is a vertex of P. Once a vertex of P becomes a vertex of Q, it can never become a non-vertex of Q again, therefore this case can happen at most 2024 times as A' and C' weren't vertices of Q before this query.

To summarize, our algorithm is as follows: Pick a bad vertex B. Query for a line parallel to AC. If we get a line passing through B (orange) or through AC (blue), there is nothing else to do. Otherwise, pick a new bad B and repeat. All cases can only occur at most 2024 times each, therefore the algorithm stops and all vertices of Q are good at this point. Then $P = Q$.

Note: When applying the algorithm we do not know if the current answer is case 3 or case 4. This solution proves that $n = 4 \cdot 2025$ suffices. □

Solution 2. We present an alternative argument that cases 3 and 4 from the above solution can happen finitely many times.

Note that in both cases the number of the sides of Q is increased by one.

We can observe that the maximum number of sides of Q is at most 4048, because each side of Q contains a vertex of P and each vertex of P can be part of at most two sides of Q. This means that cases 3 or 4 can occur at most 4048 times in a row, as otherwise the number of sides of Q would increase by at least 4049 times in a row. Thus before the first case 1 or case 2, and between any case 1 and case 2, and after the last case 1 or case 2, at most 4048 steps can be taken. Thus the algorithm stops in at most $4048 \cdot (2024 + 2024 + 1)$ steps and we found $P = Q$ as above. □

Solution 3. We use the same observation as the previous solution. If 3 answer lines intersect at a common point X, then X must be a vertex of P. Start by querying the 4 sides of the paper. Define Q, good vertices and bad vertices as above. In this solution we use the pigeonhole principle with the observation to find vertices of P.

We will prove that we can always find a new vertex in at most $2024 \cdot 3 + 2$ steps if we haven't found all vertices of P yet. Hence we find P in at most $n = 4 + 2024 \cdot (2024 \cdot 3 + 2)$ steps. (Note that we could give a much better n with a bit more care.)

Assume that we have found $k < 2024$ vertices of P so far. By design, P lies in Q and both are convex, so if all of Q's vertices were good, $P = Q$ and we are done. Hence there is at least one bad vertex of Q.

In case there are two bad neighboring vertices A and B of Q, and let C be the next vertex of Q after A and B in this order. As the line AB is a side of Q, it must be an answered line, so it contains a vertex of P. Furthermore, in this line, only the points of segment AB are contained in Q, hence P has a vertex on segment AB. Choose an arbitrary point C' inside the segment BC, and let us query a line parallel to AC' outside the paper in half-plane AC' containing B. It is easy to see that the answer we get must intersect segment AB (possibly going through one of its endpoints), and it cannot pass through any other vertex of Q. Therefore it cannot pass through a previously known good vertex. Hence with $2024 \cdot 2 + 1$ different such queries (always choosing a different point C' from BC), by the pigeonhole principle there must be a vertex of P with at least 3 of these answered lines passing through it, and so we found a new good vertex. So in this case, we find a new vertex in at most $2024 \cdot 2 + 1$ many queries.

If there are no neighboring vertices of Q which are both bad, then there must be neighboring vertices A, B, C with A and C being good, and B being bad. Let us query a line parallel to AC outside the paper in half-plane AC containing B. We know that the boundary line of the answer intersects Q, and the given half-plane contains A and C, hence there are 3 options.

If the boundary line goes through B, then it is a new good vertex as this is the third answer line going through it. In this case we immediately found a new good vertex, so in this case, we find a new vertex in 1 query.

If the boundary line doesn't go through B, and also doesn't go through A and C, then it must intersect the segments AB and BC at some points A' and C. Then A' and C' become neighboring bad vertices of the polygon obtained by the intersection of Q with this half-plane, and so we can apply the previous case to find a new vertex of P in at most $2024 \cdot 2 + 1$ queries. So in this case, we find a new vertex in at most $2024 \cdot 2 + 2$ many queries.

Finally, if the boundary line is AC, then the intersection of Q and this half-plane has one less side. This case can happen at most 2024 times, as in this case we find a side of P. Hence, after at most 2024 steps we can apply one of the previous cases finding a new good vertex to find a new vertex of P. So in this case, we find a new vertex in at most $2024 \cdot 3 + 2$ many queries.

This finishes the proof. $\square$

Solution 4. Let $m = 2024 \cdot 2 + 1$. Then, by the pigeonhole principle, if we query m pairwise non-parallel lines, there is a vertex of P through which at least 3 of the m lines pass through. Call a vertex of P found if we have already received at least 3 answer lines passing through it.

We start by asking any m pairwise non-parallel lines outside the paper and asking m lines parallel to the first m lines so that the paper is in the strips for each pair of parallel lines. Then there is a vertex A_1 of P through which at least 3 answered lines go. However, of each parallel line pair, at most one can go through A_1, so at least m answered lines do not go through A_1. Then there is a vertex $A_2 \neq A_1$ of P through which at least 3 of the answered lines go through. So we know that A_1 and A_2 are vertices of P.

Now suppose we already have found vertices $A_1, A_2, ..., A_k$ of P forming convex polygon P' so that they are in this order on the boundary of P', and suppose that $k < 2024$. We will show that we can find a new vertex of P in a bounded number of queries.

We first query a line parallel to A_1A_2 outside the paper so that P' and the queried line fall on different sides of line A_1A_2. This tells us either that A_1A_2 is edge of P, or gives us a line l parallel to A_1A_2 containing a vertex of P, which we have not found yet.

When we get the line l, we choose m different points $(X_i)_{1\leq i\leq m}$ on the perpendicular bisector of A_1A_2 such that they lie in between l and A_1A_2 and all the points of P' lie on the same side of lines A_1X_i and A_2X_i for all i. We query m lines parallel to A_1X_i outside the paper so that they are closer to A_1 than A_2 and m lines outside the paper parallel to X_iA_2 so that they are closer to A_2 than A_1.

Notice that from each pair of answered lines (parallel to A_1X_i and X_iA_2 respectively) at least one must not pass through any found vertex, as the only vertex of P the answers can go through are A_1 and A_2, so otherwise line l could not touch P. Thus there are at least m of the $2m$ answered lines not passing through any vertex of P', hence we find a new vertex.

By repeating the steps above, the first case of finding a side of P can happen at most 2024 times, and if it does not happen, we find a new vertex in at most $1 + 2m$ queries. Therefore we find P in at most $n = m + m + 2024 + 2022 \cdot (1 + 2m)$ many steps. $\square$