Preparation for EMC 2024

Third Training Test for Senior Category

8th December 2024

Problem 1. An infinite sequence $a_1, a_2, ...$ of positive integers is such that $a_n \geq 2$ and a_{n+2} divides $a_{n+1} + a_n$ for all $n \geq 1$. Prove that there exists a prime which divides infinitely many terms of the sequence.

Problem 2. Let $\mathbb{R}^+$ be the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x, y \in \mathbb{R}^+$,

$$f(xy + f(x)) = xf(y) + 2\,.$$

Problem 3. Let ABC be a triangle with $\angle BAC = 60°$; AD, BE, and CF be its bisectors; P, Q be the projections of A to EF and BC respectively; and R be the second common point of the circle DEF with AD. Prove that P, Q, R are collinear.

Problem 4. here is a sheet of paper (like this one) on an infinite blackboard. Marvin secretly chooses a convex $2024-$ gon P that lies fully on the piece of paper. Tigerin wants to find the vertices of P. In each step, Tigerin can draw a line g on the blackboard that is fully outside the piece of paper, then Marvin replies with the line h parallel to g that is the closest to g which passes through at least one vertex of P. Prove that there exists a positive integer n such that Tigerin can always determine the vertices of P in at most n steps.

Allotted time: 4 hours.