Preparation for EMC 2024

Second Training Test for Junior Category

Solutions

Problem 1. Two players play a game where they take turns to make a move and which always finishes with the victory of one of the players. The game is also designed so that it ends in at most n steps, for some fixed positive integer n. Prove that either the first or the second player has a Winning strategy.

Solution. We prove the statement by induction on the number n of steps after which the game certainly finishes. Let us call the two players A and B and we assume that A makes the first move.

If $n = 1$, then certainly one of the players has a winning strategy. Indeed, only player A will get to move. If any of his moves leads to him winning, then a winning strategy for player A is to make this move. If every one of his moves leads to a win for B, then a Winning strategy for B is to let A move arbitrarily.

Assume now that the result holds for some $n \geq 1$. We shall prove that it also holds for $n + 1$.

After any move by player A we are left with a game that certainly ends in at most n steps. Hence by the inductive hypothesis one of the two players has a winning strategy. If any move by player A leaves a position in which player A has a winning strategy, then making this move and then following that winning strategy is a winning strategy for player A. Otherwise, it must be that whatever first move player A makes, we are left in a situation where player B has a winning strategy. Then a winning strategy for player B is to let A make his first move arbitrarily and then follow the resulting winning strategy. This completes our induction step. □

Remark. The same argument which we used in the induction step applies to prove the more general result where we only require that the game finishes in a finite number of steps, not necessarily bounded by a fixed n.

Problem 2. Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of BC from A. Let L be the reflection of K in the line AB, and let M be the reflection of K in the line BC. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM and BC are concurrent.

Solution. Let H_A and H_C be the reflections of H across BC and BA, respectively; it is well-known that the points H_A and H_C lie on Γ. Let E' be the second intersection of line H_AM with Γ. By construction, lines $E'M$ and HK concur on BC, and our goal is to show that B, L, E', M are concyclic.

First, we claim that L, HC, and E' are collinear. Due to the reflections,

$$\angle\overline{LH_CB} = -\angle\overline{KHB} = \angle\overline{MH_AB} = \angle\overline{E'H_AB} = \angle\overline{E'H_CB}\,,$$

(the notation $\angle\overline{XYZ}$ stands for a directed angle) which proves the claim. Then

$$\angle\overline{LE'M} = \angle\overline{H_CE'H_A} = \angle\overline{H_CBH_A} = 2\angle\overline{ABC}$$

(the last equality following from reflections; verify it yourself) and

$$\angle\overline{LBM} = \angle\overline{LBK} + \angle\overline{KBM} = 2\angle\overline{ABK} + 2\angle\overline{KBC} = 2\angle\overline{ABC}\,,$$

so B, L, E', M are concyclic. Hence $E = E'$ and we are done. □

Problem 3. Prove that every positive integer n is a sum of one or more numbers of the form 2^r3^s, where r and s are non-negative integers and no summand divides another (for example, $23 = 9 + 8 + 6$).

Solution. We argue by contradiction. Suppose n is the smallest positive integer without a *proper representation*. Clearly, $n \geq 7$. Moreover, n is odd. Indeed, if n were even, then $\frac{n}{2}$ can be properly represented; by multiplying each of the powers in a proper representation of $\frac{n}{2}$ by 2, we get a proper representation of n.

Let k be such that $3^k \leq n < 3^{k+1}$. If $n = 3^k$ a proper representation is clear. If $n > 3^k$, then $\frac{n-3^k}{2}$ is an integer which can be represented as a desired sum. Multiply each term in a proper representation of $\frac{n-3^k}{2}$ by 2 and add a summand 3^k. We claim we get a proper representation for n.

Clearly no summand coming from the used proper representation of $\frac{n-3^k}{2}$ can divide another such summand. Also since all such summands are even they cannot divide the summand 3^k. Since $\frac{n-3^k}{2}$ is less than 3^k, no summand coming from the representation of $\frac{n-3^k}{2}$ can be divisible by 3^k. □

Problem 4. If $0 < a \le b \le c \le d$, prove that

$$a^b b^c c^d d^a \ge b^a c^b d^c a^d \,.$$

Solution. We shall use the following lemma.

Lemma. If a real function f is convex on the interval I and $x, y, z \in I$, $x \le y \le z$, then

$$(y-z)f(x) + (z-x)f(y) + (x-y)f(z) \le 0\,.$$

Proof. The inequality is obvious for $x = y = z$. If $x < z$, then there exist non-negative p, r such that $p + r = 1$ and $y = px + rz$. Then by Jensen's inequality $f(px + rz) \le pf(x) + rf(z)$, which is equivalent to the statement of the lemma. $\diamond$

By applying the lemma to the convex function $-lnx$ we obtain $x^y y^z z^x \ge y^x z^y x^z$ for any $0 < x \le y \le z$. Multiplying the inequalities $a^b b^c c^a \ge b^a c^b a^c$ and $a^c c^d d^a \ge c^a d^c a^d$ we get the desired inequality. $\square$

Remark. Similarly, for $0 < a_1 \le a_2 \le \cdots \le a_n$ it holds that

$$a_1^{a_2} a_2^{a_3} \cdots a_n^{a_1} \ge a_2^{a_1} a_3^{a_2} \cdots a_1^{a_n}\,.$$