Preparation for EMC 2024

Second Training Test for Senior Category

Solutions

Problem 1. Let n be a positive integer and $x_1, x_2, \ldots, x_n$ are positive real numbers. For every positive integer k we denote the sum $x_1^k + x_2^k + \cdots + x_n^k$ with S_k.
a) Prove that $S_1 < S_2$ implies $S_k < S_{k+1}$, for every positive integer k.
b) For which n there exist x_i, $i \in \{1, 2, \ldots, n\}$ such that $S_1 > S_2$ and $S_k < S_{k+1}$ for every $k > 2$?

Solution. a) Fore every integer m and positive real number x we have

$$x^m(x-1)^2 \geq 0 \implies x^{m+2} - x^{m+1} \geq x^{m+1} - x^m.$$

Adding these equations for all x_i, $i \in \{1, 2, \ldots, n\}$ we get $S_{m+2} - S_{m+1} \geq S_{m+1} - S_m$. This generalizes to

$$S_{k+2} - S_{k+1} \geq S_{k+1} - S_k \geq S_k - S_{k-1} \geq \cdots \geq S_3 - S_2 \geq S_2 - S_1 > 0,$$

which is the required claim. □
b) Since for $0 < x < 1$, $x > x^2 > x^3 > \ldots$ holds, this claim can't be true for $n = 1$. Let $n > 1$. From a) we can conclude that it is enough to find numbers for which $S_1 > S_2$ and $S_2 < S_3$. Besides that we will need some of the numbers to be less than 1 (for $x > 1$, $x^k > x^{k-1}$ implies $S_{k+1} > S_k$). Let $x_1 = t$, $x_2 = \frac{1}{2}$, and $x_3 = \cdots = x_n = 1$. We have $S_1 = t + n - 2 + \frac{1}{2}$, $S_2 = t^2 + n - 2 + \frac{1}{4}$, and $S_3 = t^3 + n - 2 + \frac{1}{8}$. Hence we need real number t such that $t^2 - t < \frac{1}{4}$ and $t^3 - t^2 > \frac{1}{8}$. One such number is $t = \frac{6}{5}$, since

$$t^2 - t = \frac{36 - 30}{25} = \frac{6}{25} < \frac{1}{4}, \text{ and}$$

$$t^3 - t^2 = \frac{36(6-5)}{125} = \frac{36}{125} > \frac{1}{8}.$$

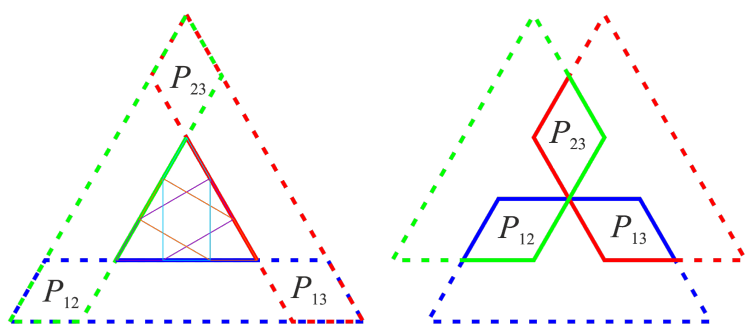
Hence the claim is true for every $n \geq 2$. □

Problem 2. Let S be a square of side 9. Inside it there are 6 convex polygons, each with area 16. Prove that at least two of the polygons have intersection polygon (polygon that is inside both of them), that have area grater than 1. Is the claim still true without the convexity condition?

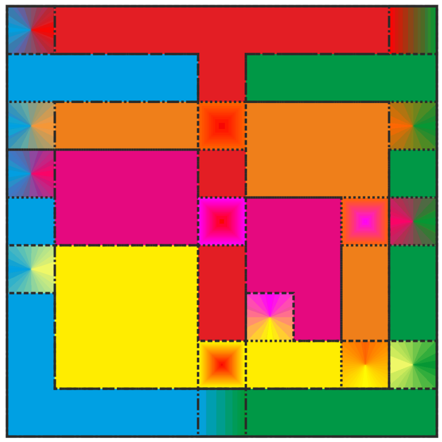
Solution. Let us suppose that there are 6 polygons, P_1, P_1, P_3, P_4, P_5, and P_6, each of area 16, such that no two of them have intersection with area greater than 1. Using the inclusion/exclusion principle we get:

$$81 = 9^2 \geq \text{area}(P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6) \geq$$
$$\sum_{i=1}^{6} \text{area}(P_i) - \sum_{i=1}^{5} \sum_{j=i+1}^{6} \text{area}(P_i \cap P_j) \geq 6 \cdot 16 - 15 \cdot 1 = 81.$$

This proves that for $i \neq j$, $\text{area}(P_i \cap P_j) = 1$ and for $i \neq j \neq k \neq i$, $\text{area}(P_i \cap P_j \cap P_k) = 0$.
For $P_{ij} = P_i \cap P_j$ we have two cases for positioning of P_1, P_2, and P_3 shown bellow with middle "triangle" inside - left, and without - right (the polygons need not be exactly as in the picture).



However we can always chose them to be like in the picture in the left (just change P_1 with a polygon that has a point in the lower part of P_1 which will imply that P_{12}, P_{13} or both will be inside the "triangle" in the middle). We furthermore chose them such that the middle "triangle" have maximal area, which implies that the other three polygons pass through the middle "triangle" (otherwise we can use this polygon and increase the area).
Now consider the borders of P_1, P_2, and P_3 that form the middle "triangle" (in solid line). Each of them has to be covered by the other three polygons. Furthermore each polygon covers a distinct segment. Since the polygons are convex none of them can cover two segments containing different vertices of the middle "triangle" (common points for two of P_1, P_2 and P_3), hence they must be divided as in the picture. This implies that $\text{area}(P_3 \cap P_4 \cap P_5) > 0$, which is a contradiction.



On the picture above it is shown that the clam need not hold for non-convex polygons. □

Problem 3. For which positive integers n there exist integer m such that $7^n \mid (5^m + 3^m - 1)$?

Solution. We will prove that $m = 7^{n-1}$ satisfies the divisibility relation. First we prove $7^k \mid 3^{7^{k-1}} + 4^{7^{k-1}}$. This is obviously true for $k = 1$ $(7 \mid 3^1 + 4^1)$. If this is true for $k = n$, then for $k = n + 1$ and $s = 7^{n-1}$ we have

$$(3^s)^7 + (4^s)^7 = (3^s + 4^s)((3^6)^s + (3^5(-4))^s +$$
$$(3^4(-4)^2)^s + (3^3(-4)^3)^s + (3^2(-4)^4)^s + (3(-4)^5)^s + ((-4)^6)^s),$$

which is divisible by 7^{n+1}, since $7^n \mid 3^s + 4^s$, $-4 \equiv 3 \pmod 7$, and we have seven congruent numbers modulo 7 in the right bracket.
Similarly $7^k \mid 5^{7^{k-1}} + 2^{7^{k-1}}$, since

$$(5^s)^7 + (2^s)^7 = (5^s + 2^s)((5^6)^s + (5^5(-2))^s +$$
$$(5^4(-2)^2)^s + (5^3(-2)^3)^s + (5^2(-2)^4)^s + (5(-2)^5)^s + ((-2)^6)^s).$$

Now we have $3^m + 5^m - 1 \equiv -(2^{2m} + 2^m + 1) \pmod{7^n}$. Multiplying by $1 - 2^m$, we get $(1 - 2^m)(3^m + 5^m - 1) \equiv 2^{3m} - 1 \equiv 8^m - 1 \pmod{7^n}$. As before we have $7^n \mid 8^m - 1$ since

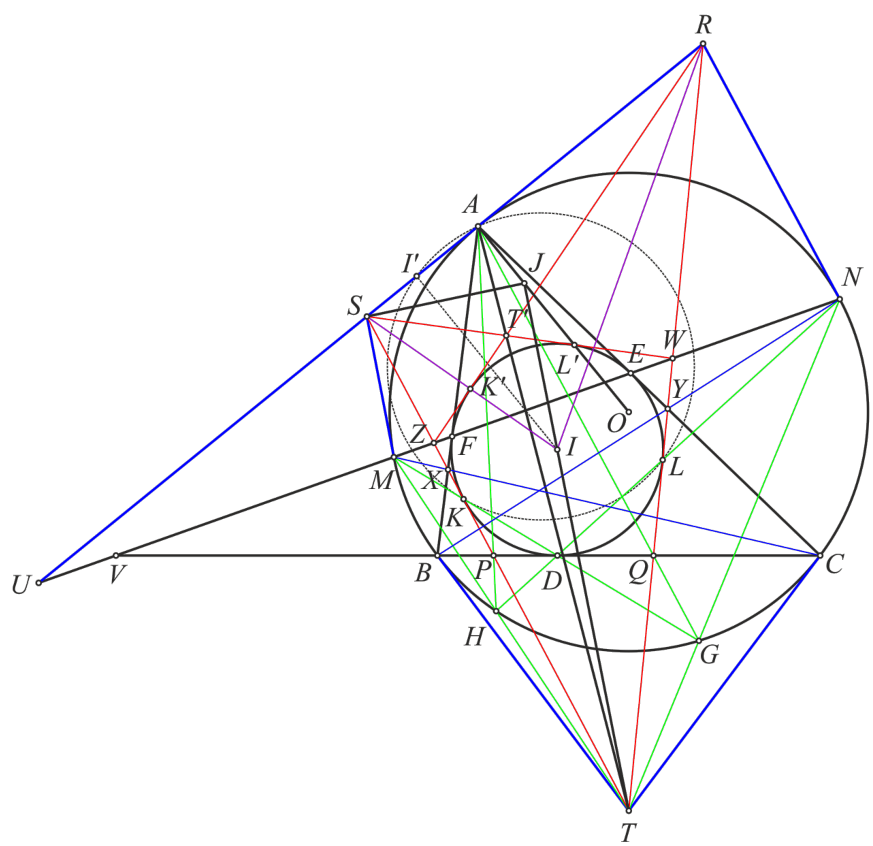
$$(8^s)^7 - 1 = (8^s - 1)((8^6)^s + (8^5)^s + (8^4)^s + (8^3)^s + (8^2)^s + 8^s + 1),$$

implying $7^n \mid (1 - 2^m)(3^m + 5^m - 1)$. Finally $7 \mid 2^k - 1$ only when $3 \mid k$. Since $3 \nmid m$ we conclude that $7^n \mid 3^m + 5^m - 1$. $\square$

Comment. The conclusions $7^n \mid a^m + b^m$ for $7 \mid a + b$ can be devised easier by using Lifting The Exponent Lemma.

Problem 4. Let $\triangle ABC$ be scalene and acute-angled with incircle with centre I that touches the sides BC, CA, and AB in D, E, and F, respectively. Let k with centre O be the circumcircle of $\triangle ABC$ and the ray EF intersects k at M. The tangents of k at A and M meet at S, and the tangents at B and C meet at T. If IT intersects OA at J prove that $\angle ASJ = \angle TSI$.

Solution. Let ray FE intersect k at N, the tangents of k at A and N meet at R, MD and ND intersect k for the second time in G and H respectively, and the incircle of $\triangle ABC$ at K and L respectively, AH and AG intersect BC at p and Q respectively and MN meets BC at V.



Since AD, BE and CF are concurrent, $-1 = (V, D; B, C) \stackrel{N}{=} (M, H; B, C)$, hence H lies on MT. And by reason of symmetry G lies on NT. Also $-1 = (AM, AH; AB, AC) = (AM, AH; AE, AF)$ implies that AH is polar line of M in respect to the incircle, hence KP is tangent to the incircle

($M-K-D$ is polar line of P). And similarly LQ is also tangent to the incircle.
If KP meets AB at X, and LQ meets BC at Y, then M, X, and C are collinear (they are on the polar line of the intersection of FK and ED) and N, Y, and B are collinear (they are on the polar line of the intersection of EL and FD).
Applying Pascal theorem on $BHCMBA$ we conclude that X, P, and T are collinear, and similarly Y, Q, and T are collinear. Pints S, K, and T lie on the polar line of the intersection of AM and BC with respect to k. Hence, we conclude that TS is tangent to the incircle at K, and similarly TR is tangent to the incircle at L.
Let TS meet MN at Z and AK meets the incircle for the second time at K'. Since MS, SR, and RN are tangent to k we have $(U,A;S,R)=-1=(ZK',ZK;ZA,ZE)$, implying that Z, K', and R are collinear and lie on the tangent of the incircle at K'. Similarly if TR meet MN at W and AL meets the incircle for the second time at L', the line $W-L'-S$ is tangent to the incircle at L'.
If T' is the intersection of ZR and WS, then using $(U,A;S,R)=-1$ we conclude that A, T, and T' are collinear. Consider the pedal circle of I in $\triangle TSR$ passing through K, L and I' - the feet of the altitude form I on RS. We have

$$\angle KAL = \angle KAT + \angle TAL = \angle K'KS - \angle ATK + \angle RLL' - \angle LTA = \frac{1}{2}(\angle K'ZS - \angle LTK) + \frac{1}{2}(\angle RWL' - \angle LTK) = \frac{1}{2}\angle ZRL + \frac{1}{2}\angle KSW = \angle KSI + \angle IRL = \angle KI'I + \angle II'L = \angle KI'L,$$

implying that A lies on the petal circle. Since $AO \perp RS$, the isogonal conjugate of I lies on AO and since TI is the angle bisector, the isogonal conjugate is J. Hence $\angle ASJ = \angle TSI$. □