Preparation for EMC 2024

First Training Test for Junior Category

Solutions

Problem 1. Determine whether there is a natural number n for which $8^n + 47$ is prime.

Solution. The number $m = 8^n + 47$ is never prime.

If n is even, say $n = 2k$, then $m = 64^k + 47 \equiv 1 + 2 \equiv 0 \pmod{3}$. Since also $m > 3$, then m is not prime.

If $n \equiv 1 \pmod{4}$, say $n = 4k + 1$, then $m = 8(8^k)^4 + 47 \equiv 3 + 2 \equiv 0 \pmod{5}$. Since also $m > 3$, then m is not prime.

If $n \equiv 3 \pmod{4}$, say $n = 4k + 3$, then

$$m = 8(64^{2k+1} + 1) \equiv 8((-1)^{2k+1} + 1) \equiv 0 \pmod{13}.$$

Since also $m > 13$, then m is not prime. □

Problem 2. Let $a, b, c, d > 0$ such that $a + b + c + d = 1$. Prove the inequality:

$$\frac{1}{4a + 3b + c} + \frac{1}{3a + b + 4d} + \frac{1}{a + 4c + 3d} + \frac{1}{4b + 3c + d} \geq 2.$$

Solution. Let $A = \frac{1}{4a+3b+c}$, $B = \frac{1}{3a+b+4d}$, $C = \frac{1}{a+4c+3d}$ and $D = \frac{1}{4b+3c+d}$.

Using AM-HM for A, B, C, D we get:

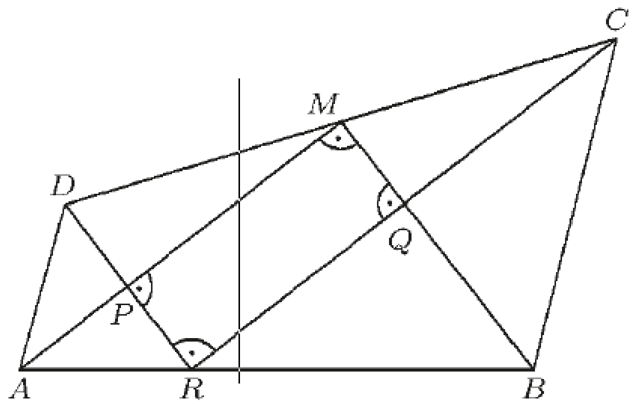
$$\frac{A + B + C + D}{4} \geq \frac{4}{\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}} \iff A + B + C + D \geq \frac{16}{\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}}.$$

$$\begin{aligned} LHS &\geq \frac{16}{(4a + 3b + c) + (3a + b + 4d) + (a + 4c + 3d) + (4b + 3c + d)} \\ &= \frac{16}{8a + 8b + 8c + 8d} \\ &= \frac{16}{8} \\ &= 2\,. \end{aligned}$$

□

Problem 3. Let $ABCD$ be a convex quadrilateral and let M be the midpoint of the side CD. Lines BM and AM are perpendicular. Prove that lines BC and AD are parallel if and only if, $AB = BC + AD$.

Solution.



Let $DP \perp AM$, $P \in AM$, $CQ \perp BM$, $Q \in BM$ and $DP \cap CQ = \{R\}$.

The quadrilateral $PRQM$ is a rectangle and MP and MQ are middle segments in $\triangle DRC$. Hence $CQ = QR$, $DP = PR$ and we conclude that $\triangle ADR$ and $\triangle BCR$ are both isosceles. Therefore

$$AR = AD, \quad BR = BC \tag{1}$$

and

$$\angle ARB = \angle ARD + \angle DRC + \angle CRB = \angle ADR + 90^\circ + \angle BCR. \tag{2}$$

Suppose $AB = AD + BC$. It follows from (1) that

$$AD + BC = AR + RB \geq AB = AD + BC$$

and we conclude that $R \in AB$. Thus, $\angle ARB = 180^\circ$ and (2) implies $\angle ADR + \angle BCR = 90^\circ$. Therefore

$$\angle ADC + \angle BCD = \angle ADR + \angle BCR + \angle RDC + \angle RCD = 180^\circ$$

giving that $AD \parallel BC$.

Suppose $AD \parallel BC$. We have $\angle ADC + \angle BCD = 180^\circ$ and since $\angle RDC + \angle RCD = 90^\circ$ we conclude that $\angle ARD + \angle BRC = 90^\circ$. It follows by (2) that $\angle ARC = 180^\circ$ and therefore

$$AB = AR + RB = AD + BC.$$

□

Problem 4. Consider a 2×2011 table. Two players in turn place dominoes on it, the first one places only horizontal dominoes and the second one places only vertical dominoes. The dominoes may not overlap. The player who has no legal move loses the game. Which player has a winning strategy?

Solution. Denote the player who places horizontal dominoes by A and the player who places vertical ones by B. We prove that A has a winning strategy We show first that if it is player B's move on an empty 2×4 table then he loses the game. Indeed, observe that no matter where B places his vertical domino then A always has a move. It is easy to see now that irrelevant to the move of A, B has a unique move after which A also has a move. Now the table is covered and player A wins.

Consider now a 2×2011 table. Let A split the table into 502 tables 2×4 and one table 2×3. His first move is to put a domino anywhere in the 2×3 table. Note that the vertical dominoes of B lie entirely inside one of the small tables.

Player A now follows the strategy: Each time B puts a domino in one of the 2×4 tables then A places a domino in the same 2×4 table. According to the above observation this is always possible.

If B puts his vertical domino in the 2×3 table then A puts another horizontal domino in this table. Thus the 2×3 table is completely covered and no other moves in this table are possible The described strategy shows that no matter what moves B makes, A always has a move Thus the last move will belong to A. Therefore the first player has a winning strategy. $\square$