Preparation for EMC 2024

First Training Test for Senior Category

Solutions

Problem 1. We are given a set of 2024 distinct points in the plane, no three collinear. Four points from this set are vertices of a unit square; the other 2020 points lie inside this square. Prove that there exist three distinct points X, Y, Z in this set such that $P_{\triangle XYZ} \leq \frac{1}{4042}$.

Solution. We prove by induction on n that, given $n \geq 1$ points inside the square (with no three collinear), the square may be partitioned into $2n + 2$ triangles, where each vertex of these triangles is either one of the n points or one of the vertices of the square. For the base case $n = 1$, because the square is convex, we may partition the square into four triangles by drawing line segments from the interior point to the vertices of the square.

For the induction step, assume that the claim holds for some $n \geq 1$. Then for $n + 1$ points, take n of the points and partition the square into $2n+2$ triangles whose vertices are either vertices of the square or are among the n chosen points. Call the remaining point P. Because no three of the points in the set are collinear, P lies inside one of the $2n + 2$ partitioned triangles, say inside $\triangle ABC$. We may further divide this triangle into the triangles APB, BPC and CPA. This yields a partition of the square into $2(n + 1) + 2 = 2n + 4$, completing the induction.

For the special case $n = 2020$, we may divide the square into 4042 triangles with total area 1. One of those triangles has area at most $\frac{1}{4042}$, as desired. $\square$

Problem 2. Find all $f : \{1, 2, 3, \ldots\} \to \{1, 2, 3, \ldots\}$ such that for every $m, n \in \{1, 2, 3, \ldots\}$ holds

$$f(m) + f(n) \mid m + n\,.$$

Solution. We will prove by strong induction that $f(n) = n$. Letting $m = n = 1$, we obtain that $2f(1) \mid 2$, hence $f(1) \mid 1 \Rightarrow f(1) = 1$. This establishes the base case. Assume now that the result holds for all positive integers which are less than some $n > 1$. By Chebyshev's theorem, we know that there exists a prime number between n and $2n$, so there exists $m < n$ such that $m + n = p$ is prime. Since $f(m) + f(n)$ divides p, we have that $f(m)+f(n)$ is either 1 or p. But $f(m)+f(n) \geq 1+1 = 2$, so it cannot be 1. Therefore $f(m)+f(n) = p$. Since $f(m) = m$ from the induction hypothesis, we obtain $f(n) = p - m = n$. This completes our proof. $\square$

Problem 3. Let $a_0, a_1, a_2, \ldots$ be a strictly increasing sequence of non-negative integers such that every non-negative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j and k are not necessarily distinct. Determine all possible values of a_{2024}.

Solution. First we will prove that this sequence is unique, showing that a_n is uniquely determined, by induction on n. Clearly $a_0 = 0$, so the base case holds. Now, suppose we have proven that

$$\{a_0, a_1, \ldots\} \cap \{1, 2, \ldots, n\}$$

is uniquely determined. If $n+1$ can be written as $x+2y+4z$ where $x, y, z \in \{a_0, a_1, \ldots\} \cap \{1, 2, \ldots, n\}$ then it cannot belong to the sequence due to the uniqueness of this representation, while if not, it definitely must because such a representation should exist. So, whether $n+1$ belongs or not to the sequence depends only on the terms of the sequence smaller that $n+1$. As these terms are uniquely determined, the induction step is done.

Therefore, it suffices to find an example of such a sequence. The expression $a_i + 2a_j + 4a_k$ is strongly related to base 8 expansion and with this idea in mind, we easily find that the sequence consists of those non-negative integers whose digits in base 8 are only 0 or 1. Then we check that a_n is obtained by writing n in base 2 and reading the result in base 8. In particular,

$$2024 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3$$

so

$$\begin{aligned} a_{2024} &= 8^{10} + 8^9 + 8^8 + 8^7 + 8^6 + 8^5 + 8^3 \\ &= 1,227,129,344\,. \end{aligned}$$

□

Problem 4. Each of the numbers $1, 2, \ldots, N$ is colored black or white. We are allowed to simultaneously change the colors of any three numbers in arithmetic progression. For which numbers N can we always make all the numbers white?

Solution. We clearly cannot always make all the numbers white if $N = 1$. Suppose that $2 \le N \le 7$, and suppose that only the number 2 is colored black. Call a number from $\{1, \ldots, N\}$ heavy if it is not congruent to 1 modulo 3. Let X be the number of heavy numbers which are black, where X changes as we change the colors. Suppose we change the colors of the numbers in $\{a - d, a, a + d\}$, where $1 \le a - d < a < a + d \le N$. If d is not divisible by 3, then $a - d, a, a + d$ are all distinct modulo 3, so exactly two of them are heavy. If instead d is divisible by 3, then $a - d, a, a + d$ must equal $1, 4, 7$, none of which are heavy. In either case, changing the colors of these three numbers changes the color of an even number of heavy numbers. Hence X is always an odd number, and we cannot make all the numbers white.

Next we show that for $N \ge 8$, we can always make all the numbers white. To do this, it suffices to show that we can invert the color of any single number n. We prove this by strong induction. If $n \in \{1, 2\}$, then we can invert the color of n by changing the colors of the numbers in $\{n, n+3, n+6\}$, $\{n + 3, n + 4, n + 5\}$ and $\{n + 4, n + 5, n + 6\}$. Now assuming that we can invert the color of $n - 2$ and $n - 1$ (where $3 \le n \le N$), we can invert the color of n by first inverting the colors of $n - 2$ and $n - 1$ then changing the colors of the numbers in $\{n - 2, n - 1, n\}$.

Hence, we can always make all the numbers white if and only if $N \ge 8$. $\square$