Preparation for EMC 2024

First Training Test for Senior Category

24th November 2024

Problem 1. We are given a set of 2024 distinct points in the plane, no three collinear. Four points from this set are vertices of a unit square; the other 2020 points lie inside this square. Prove that there exist three distinct points X, Y, Z in this set such that $P_{\triangle XYZ} \leq \frac{1}{4042}$.

Problem 2. Find all $f : \{1, 2, 3, \ldots\} \to \{1, 2, 3, \ldots\}$ such that for every $m, n \in \{1, 2, 3, \ldots\}$ holds

$$f(m) + f(n) \mid m + n\,.$$

Problem 3. Let $a_0, a_1, a_2, \ldots$ be a strictly increasing sequence of non-negative integers such that every non-negative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j and k are not necessarily distinct. Determine all possible values of a_{2024}.

Problem 4. Each of the numbers $1, 2, \ldots, N$ is colored black or white. We are allowed to simultaneously change the colors of any three numbers in arithmetic progression. For which numbers N can we always make all the numbers white?

Allotted time: 4 hours.