Preparation for EMC 2023

First Training Test for Senior Category

19th November 2023

Problem 1. The sequence $a_1, a_2, a_3, \dots$ is defined by: $a_1 = 1$ and $a_{n+1} = a_n^2 + 1$ for $n \geq 1$. Prove that there exists a positive integer n such that a_n has a prime factor with more than 2023 digits.

Problem 2. Each square in a 2023×2023 grid of unit squares can be colored either red or blue. We can adjust the colors of the squares with a sequence of moves. In each move, we choose a rectangle composed of unit squares, and change all of its red squares to blue and all of its blue squares to red. A *monochrome path* in the grid is a sequence of distinct unit squares of the same color, such that each shares an edge with the next. A coloring of the grid is called *tree-like* if, for any two unit squares S and T of the same color, there is a unique monochrome path whose first square is S and last square is T.

Determine the minimum number of moves required to reach a tree-like coloring when starting from a coloring in which all unit squares are red.

Problem 3. Let $\triangle ABC$ be a triangle with incenter I. Suppose that D is a variable point on the circumcircle of $\triangle ABC$, on the arc $\widehat{AB}$ that does not contain C. Let E be a point on the line segment BC such that $\angle ADI = \angle IEC$. Prove that, as D varies, the line DE passes through a fixed point.

Problem 4. Prove that for each integer k satisfying $2 \leq k \leq 100$, there are positive integers $b_2, b_3, \dots, b_{101}$ such that

$$b_2^2 + b_3^3 + \cdots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \cdots + b_{101}^{101}.$$

Allotted time: 4 hours.