Preparation for EMC 2023

First Training Test for Senior Category

Solutions

Problem 1. The sequence $a_1, a_2, a_3, \dots$ is defined by: $a_1 = 1$ and $a_{n+1} = a_n^2 + 1$ for $n \geq 1$. Prove that there exists a positive integer n such that a_n has a prime factor with more than 2023 digits.

Solution. Call a prime p *good* if there exists a positive integer n such that a_n is divisible by p. It suffices to show that there are infinitely many good primes.

For a good prime p, let d be the smallest positive integer such that $a_d \equiv 0 \pmod{p}$. Define $a_0 = 0$. An easy induction yields $a_i \equiv a_{i+d} \pmod{p}$ for all integers $i \geq 0$. So $a_n \equiv 0 \pmod{p}$ whenever $d \mid n$.

Suppose, for the sake of contradiction, that there are only finitely many good primes $p_1, p_2, \dots, p_k$ and let d_i denote the smallest positive integer such that $a_{d_i} \equiv 0 \pmod{p_i}$. From the preceding paragraph we know that $a_n \equiv 0 \pmod{p_i}$ whenever $d_i \mid n$. Choose $n = d_1 d_2 \cdots d_k$. Hence a_n is divisible by $p_1 p_2 \cdots p_k$. Let p be a prime factor of a_{n+1}. Hence p is good and so is in the list $p_1, p_2, \dots, p_k$. But $a_{n+1} = a_n^2 + 1$ and $p \mid a_{n+1}$ and $p \mid a_n$. Thus $p \mid 1$, which is a contradiction. $\square$

Problem 2. Each square in a 2023×2023 grid of unit squares can be colored either red or blue. We can adjust the colors of the squares with a sequence of moves. In each move, we choose a rectangle composed of unit squares, and change all of its red squares to blue and all of its blue squares to red. A *monochrome path* in the grid is a sequence of distinct unit squares of the same color, such that each shares an edge with the next. A coloring of the grid is called *tree-like* if, for any two unit squares S and T of the same color, there is a unique monochrome path whose first square is S and last square is T.

Determine the minimum number of moves required to reach a tree-like coloring when starting from a coloring in which all unit squares are red.

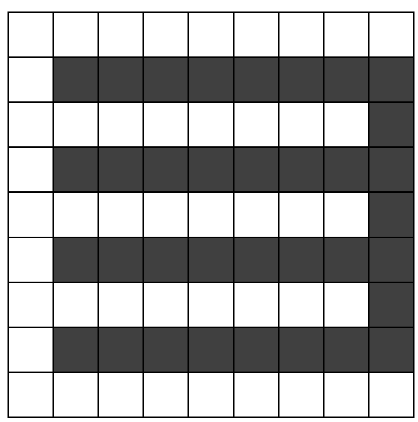
Solution. The answer for an $n \times n$ chessboard is $[\frac{n}{2}]$. So for $n = 2023$, the answer is 1011.

We first show that at least $[\frac{n}{2}]$ moves are needed. Suppose that we achieve the required state after $k \leq [\frac{n}{2}] - 1$ moves. In the chessboard there are $n-1$ interior horizontal lines and $n-1$ interior vertical lines (excluding the perimeter of the chessboard). In each move, the perimeter of the chosen rectangle is made up of two vertical and two horizontal lines. Since $2k < n-1$, after k moves, at least one vertical line, say v, and one horizontal line, say h, of the chessboard do not coincide with the perimeters of any of the k chosen rectangles. Hence the four unit squares adjacent to the intersection point of v and h have the same color after k moves. This is a contradiction since a monochrome 2×2 square cannot be part of a tree-like coloring.

It remains to show that $[\frac{n}{2}]$ moves is sufficient. Suppose the chessboard is on the Cartesian plane, described by the region $0 \leq x, y \leq n$. The required state can be achieved by the following moves.

- For the first move, choose the rectangle defined by $1 \leq x \leq n$ and $1 \leq y \leq [\frac{n}{2}]$.
- For the i^{th} move where $i = 2, 3, \ldots, [\frac{n}{2}]$, choose the rectangle defined by $1 \leq x \leq n-1$ and $2i-2 \leq y \leq 2i-1$.

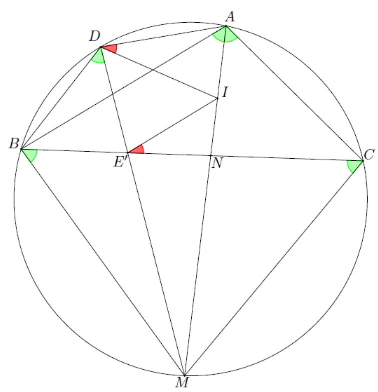
The following diagram shows the final configuration for the n = 9 case.



It is easy to check that, after $[\frac{n}{2}]$ moves, the grid indeed has a tree-like coloring. $\square$

Problem 3. Let $\triangle ABC$ be a triangle with incenter I. Suppose that D is a variable point on the circumcircle of $\triangle ABC$, on the arc $\overset{\frown}{AB}$ that does not contain C. Let E be a point on the line segment BC such that $\angle ADI = \angle IEC$. Prove that, as D varies, the line DE passes through a fixed point.

Solution. Let AI meet the circumcircle of ABC again at M. We claim M is the fixed point. Let E' be the intersection of DM and BC. We will show that $\angle ADI = \angle IE'C$, which implies $E = E'$ since $\angle IEC$ varies monotonically as E varies along BC.



First, note that $MB = MC = MI$. Certainly $MB = MC$ holds since arcs $\overset{\frown}{MB}$ and $\overset{\frown}{MC}$ subtend equal angles at the circumference. The equality $MB = MI$ follows from

$$\angle MBI = \angle MBC + \angle CBI = \angle MAC + \angle IBA = \angle MAB + \angle IBA = \angle MIB.$$

Next, let AM intersect BC at N. Since $\angle CBM = \angle MAB = \angle MDB$, we have the similarities $\triangle MBE' \ \triangle MDB$ and $\triangle MBN \ \triangle MAB$. These imply the following length conditions:

$$MI^2 = MB^2 = MD \cdot ME' = MA \cdot MN.$$

The condition $MI^2 = MD \cdot ME'$ implies that $\triangle MIE' \ \triangle MDI$, while the condition $MD \cdot ME' = MA \cdot MN$ implies that $\triangle MNE' \ \triangle MDA$. Finally, the proof can be completed by noting

$$\angle ADI = \angle ADM - \angle IDM = \angle MNE' - \angle MIE' = \angle IE'C\,.$$

Remark. The similar triangles which allowed the conversions between angle and length conditions can be replaced by power of point arguments, or an inversion with center M and radius MB = MC = MI. $\square$

Problem 4. Prove that for each integer k satisfying $2 \le k \le 100$, there are positive integers $b_2, b_3, \ldots, b_{101}$ such that

$$b_2^2 + b_3^3 + \cdots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \cdots + b_{101}^{101}.$$

Solution. Consider the equation

$$a_2^2 + \cdots + a_k^k - a_{k+1}^{k+1} - \cdots - a_{100}^{100} = L\,.$$

First of all, choose $a_2, \ldots, a_{100}$ arbitrarily so that L is positive (e.g., this can be achieved by making a_2 very big). Since 101 is coprime to 100!, there exist positive integers c and d such that $100!c + 1 = 101d$ (e.g., by setting c to be the inverse of $-100!$ in modulo 101). In fact, by Wilson's theorem, $100! + 1$ is divisible by 101, so $c = 1$ works.

Multiplying both sides by $L^{100!c}$, we have

$$(a_2 L^{\frac{100!c}{2}})^2 + \cdots + (a_k L^{\frac{100!c}{k}})^k - (a_{k+1} L^{\frac{100!c}{k+1}})^{k+1} - \cdots - (a_{100} L^{\frac{100!c}{100}})^{100} = (L^d)^{101}\,.$$

Therefore, setting $b_i = a_i L^{\frac{100!c}{i}}$ for $2 \le i \le 100$ and $b_{101} = L^d$ would satisfy the required condition. □