Preparation for EMC 2023

Fifth Training Test for Senior Category

Solutions

Problem 1. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the inequality

$$f(n+1) > f(f(n))$$

for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Solution. The only solution is the identity function. We break the argument into several steps.

Claim 1. $f(1) = \min_{n\in\mathbb{N}} f(n)$. Namely, let m be the smallest natural number such that $f(m) = \min_{n\in\mathbb{N}} f(n)$. It cannot be that $m \geq 2$ for otherwise the inequality $f(m) > f(f(m-1))$ contradicts the choice of m. The claim follows. Moreover, we have shown that $n > 1$ implies $f(n) > f(1)$. ⋄

Actually, the same line of reasoning can yield the following.

Claim 2. *f is strictly increasing.* Let us prove by induction on n that $f(x) > f(n)$ whenever $x > n$. The case $n = 1$ is the content of Claim 1. Suppose that $n \geq 2$ and that $x > k$ implies $f(x) > f(k)$ for all $k < n$. It follows that $f(x) \geq n$ holds for all $x \geq n$. Let $f(m) = \min_{x\geq n} f(x)$. If we suppose that $m > n$, then $m - 1 \geq n$ and consequently $f(m-1) \geq n$. But in this case the inequality $f(m) > f(f(m-1))$ contradicts the minimality property of m. The inductive proof is thus completed. ⋄

Claim 3. *$f(n) = n$ for all natural numbers n.* Since f is strictly increasing, clearly $f(n) \geq n$. Moreover, $f(n+1) > f(f(n))$ implies that $n + 1 > f(n)$. Hence $f(n) = n$. ⋄

It is readily seen that the identity function meets all requirements. □

Problem 2. There are 2^n words of length n over the alphabet $\{0, 1\}$. Prove that the following algorithm generates the sequence $w_0, w_1, \ldots, w_{2^n-1}$ of all these words such that any two consecutive words differ in exactly one digit.

(1) $w_0 = 00\dots0$ (n zeros).

(2) Suppose $w_{m-1} = a_1a_2\dots a_n$, $a_i \in \{0,1\}$. Let $e(m)$ be the exponent of 2 in the representation of m as a product of primes, and let $j = 1+e(m)$. Replace the digit a_j in w_{m-1} by $1-a_j$. The obtained word is w_m.

Solution. We prove by induction on n that independently of the word w_0, the given algorithm generates all words of length n. This is clear for $n=1$. Suppose now the statement is true for $n-1$, and that we are given a word $w_0 = c_1c_2\dots c_n$ of length n. Obviously, the words $w_0, w_1, \dots, w_{2^{n-1}-1}$ all have the nth digit c_n, and by the inductive hypothesis these are all words whose nth digit is c_n. Similarly, by the inductive hypothesis $w_{2^{n-1}}, \dots, w_{2^n-1}$ are all words whose nth digit is $1-c_n$, and the induction is complete. $\square$

Problem 3. Let P be a point inside $\triangle ABC$ such that

$$\angle APB - \angle C = \angle APC - \angle B\,.$$

Let D, E be the incenters of $\triangle APB$, $\triangle APC$ respectively. Show that AP, BD, and CE meet in a point.

Solution. Let X, Y, Z respectively be the feet of the perpendiculars from P to BC, CA, AB. Examining the cyclic quadrilaterals $AZPY$, $BXPZ$, $CYPX$, one can easily see that $\angle XZY = \angle APB - \angle C$ and $XY = PC\sin\angle C$. The first relation gives that $\triangle XYZ$ is isosceles with $XY = XZ$, so from the second relation $PB\sin\angle B = PC\sin\angle C$. Hence $AB/PB = AC/PC$. This implies that the bisectors BD and CD of $\angle ABP$ and $\angle ACP$ divide the segment AP in equal ratios; i.e., they concur with AP. $\square$

Problem 4. Show that the number of $0-1$ strings of length n with exactly m zeros that are followed immediately by ones equals

$$\binom{n+1}{2m+1}.$$

Solution. Given a $0-1$ string $(a_1, a_2, \dots, a_n)$, enlarge it to a string of length $n+2$ by adding $a_0 = 1$ and $a_{n+1} = 0$. Then associate with the enlarged string a corresponding one $(b_l, b_2, \dots, b_{n+l})$ by setting $b_i = 0$ if $a_i = a_{i-1}$ and $b_i = 1$ if $a_i \neq a_{i-1}$. It is easy to check that the mapping just defined is a bijection from the set of $0-1$ strings $(a_1, a_2, \dots, a_n)$ that have exactly

m zeros which are followed immediately by ones to the set of $0-1$ strings $(b_1, b_2, \ldots, b_{n+1})$ that have $2m+1$ ones. There are $\binom{n+1}{2m+1}$ strings in the latter set, so the proof is complete. $\square$