Preparation for EMC 2023

Fourth Training Test for Senior Category

Solutions

Problem 1. Ninety-one white pawns are placed on a 10×10 chessboard. Misha repaints these pawns black one at a time and puts down each repainted pawn on an empty square of the board. Prove that eventually two pawns of different colors will occupy two squares that have a common side.

Solution. Arguing by contradiction, suppose Misha can act so that two pawns of the same color never appear in adjacent squares. At any time there exists a row and a column completely filled with pawns, which, by our supposition, are of the same color; i.e., during the whole process of repainting there is a *monochromatic cross* consisting of 19 pawns. We know that at the very beginning this cross is all white and in the end it must be all black. However, this means that eventually the repainting of some single pawn will result in the appearance of the black cross on the board just after a white cross has been present. This is clearly impossible, because any two crosses on the board have at least two common squares. $\square$

Problem 2. Let A, B, and C be noncollinear points. Prove that there is a unique point X in the plane of ABC such that

$$XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2 .$$

Solution. Let A', B', C' be the points symmetric to A, B, C with respect to the midpoints of BC, CA, AB respectively. From the condition on X we have

$$XB^2 - XC^2 = AC^2 - AB^2 = A'B^2 - A'C^2 ,$$

and hence X must lie on the line through A' perpendicular to BC. Similarly, X lies on the line through B' perpendicular to CA. It follows that there is a unique position for X, namely the orthocenter of $\triangle A'B'C'$. It easily follows that this point X satisfies the original equations. $\square$

Problem 3. Let a and b be nonnegative integers such that $ab \geq c^2$, where c is an integer. Prove that there exists a natural number n and integers $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ such that

$$\sum_{i=1}^{n} x_i^2 = a, \quad \sum_{i=1}^{n} y_i^2 = b, \quad \text{and} \quad \sum_{i=1}^{n} x_i y_i = c\,.$$

Solution. We may assume $c \geq 0$ (otherwise, we may simply put $-y_i$ in the place of y_i). Also, we may assume $a \geq b$. If $b \geq c$, it is enough to take $n = a+b-c$, $x_1 = \cdots = x_a = 1$, $y_1 = \cdots = y_c = y_{a+1} = \cdots = y_{a+b-c} = 1$, and the other x_i's and y_i's equal to 0, so we need only consider the case $a > c > b$.

We proceed to prove the statement of the problem by induction on $a+b$. The case $a+b=1$ is trivial. Assume that the statement is true when $a+b \leq N$, and let $a+b = N+1$. The triple $(a+b-2c, b, c-b)$ satisfies the condition, since $(a+b-2c)b - (c-b)^2 = ab - c^2$; so by the induction hypothesis there are n-tuples $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ with the wanted property. It is easy to verify that $(x_i+y_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ give a solution for (a,b,c). $\square$

Problem 4. Let $S(x)$ be the sum of digits of positive integer x in its decimal representation. Find the smallest value of $S(1998n)$ for positive integer n.

Solution. We prove the inequality $S(1998n) \geq 27$, with the bound being sharp (e.g., S(1998)=1+9+9+8=27). The trick is to note that

$$1998n + 2n = 2000n \quad \text{and} \quad S(2n) = S(2000n)\,,$$

the latter because $2000n = 2n \cdot 1000$. We combine these with the following easy observation: if x and y are any positive integers, then

$$S(x+y) = S(x) + S(y) - 9 \cdot t(x,y)\,,$$

where $t(x,y)$ denotes the number of 'transfers' to a higher place value when summing the corresponding (i.e., equally place valued) digits of x and y (see the remark below); e.g., $t(1988, 2) = 1$ whereas $t(1998, 2) = 3$. Plugging $x = 1998n$ and $y = 2n$, we conclude that $S(1998n) = 9 \cdot t(1998n, 2n)$. So we are left with showing that $t(1998n, 2n) \geq 3$. Now, in view of the equalities $2000n = 2n \cdot 1000$ and $1998n + 2n = 2000n$, it must be that $S(1998n, 2n) \geq 3$ always holds true. $\square$

Remark. In the decimal system, every number is written with the 10 digits

$$0, 1, 2, 3, \ldots, 9\,,$$

and the value of each digit depends on its location. The right most digits are the *ones*, the second digit from the right are the *tens*, the third digit from the right are the *hundreds*, the fourth digit from the right are the *thousands* and so on. When reading a number we multiply the right most digit by $1 = 10^0$, the second digit from the right by $10 = 10^1$, the third digit by $100 = 10^2$, the fourth digit form the right by $1000 = 10^3$ and so on. We call $1, 10, 100, 1000, \ldots$ the *values* of the places of the digits. The *place value* of the n-th digit from the right is 10^{n-1}.

Suggestion. Noting that $1998 = 2 \cdot (10^3 - 1)$, generalize the statement of the problem.