Preparation for EMC 2023

Third Training Test for Senior Category

Solutions

Problem 1. Let a, b, c and d be positive real numbers with $a+b+c+d=4$. Prove that

$$\frac{ab}{a^2-\frac{4}{3}a+\frac{4}{3}}+\frac{bc}{b^2-\frac{4}{3}b+\frac{4}{3}}+\frac{cd}{c^2-\frac{4}{3}c+\frac{4}{3}}+\frac{da}{d^2-\frac{4}{3}d+\frac{4}{3}}\leq 4.$$

Solution. First we prove that $\frac{a}{a^2-\frac{4}{3}a+\frac{4}{3}}\leq\frac{a+2}{3}$. Indeed since $a^2-\frac{4}{3}a+\frac{4}{3}\geq\frac{8}{9}$ we have:

$$\begin{aligned}
\frac{a}{a^2-\frac{4}{3}a+\frac{4}{3}}\leq\frac{a+2}{3} &\qquad\Leftrightarrow\\
(a+2)(3a^2-4a+4)\geq 9a &\qquad\Leftrightarrow\\
3a^3+2a^2-4a+8\geq 9a &\qquad\Leftrightarrow\\
(3a+8)(a^2-2a+1)\geq 0 &\qquad\Leftrightarrow\\
(3a+8)(a-1)^2\geq 0. &
\end{aligned}$$

Now we calculate

$$\begin{aligned}
\frac{ab}{a^2-\frac{4}{3}a+\frac{4}{3}}+\frac{bc}{b^2-\frac{4}{3}b+\frac{4}{3}}+\frac{cd}{c^2-\frac{4}{3}c+\frac{4}{3}}+\frac{da}{d^2-\frac{4}{3}d+\frac{4}{3}}\leq\\
\frac{(a+2)b}{3}+\frac{(b+2)c}{3}+\frac{(c+2)d}{3}+\frac{(d+2)a}{3}=\\
\frac{1}{3}(ab+2b+bc+2c+cd+2d+da+2a)=\\
\frac{1}{3}(a+c)(b+d)+\frac{2}{3}(a+b+c+d)\leq\\
\frac{(a+c+b+d)^2}{12}+\frac{2}{3}(a+b+c+d)=4
\end{aligned}$$

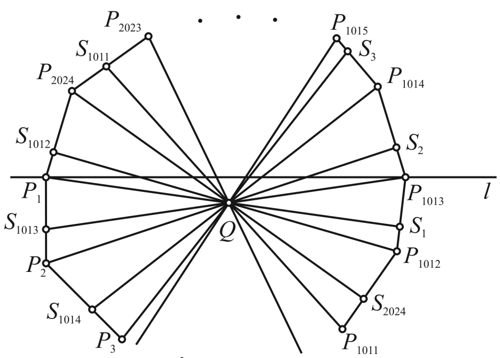
proving the inequality. □

Problem 2. Let Q be a point inside the convex polygon $P_1P_2 \cdots P_{2024}$. For each $i = 1, 2, \ldots, 2024$, extend the line P_iQ until it meets the polygon again at a point S_i. Suppose that none of the points $S_1, S_2, \ldots, S_{2024}$ is a vertex of the polygon.

Prove that there is at least one side of the polygon that does not contain any of the points $S_1, S_2, \ldots, S_{2024}$.

Solution. The diagonal $l = P_1P_{1013}$ splits the polygon into two halves. Since Q does not lie on l (otherwise $S_1 = P_{1013}$ which is not permitted), it lies strictly inside one of these halves. Without loss of generality we may assume that Q lies inside $P_1P_2 \cdots P_{1013}$.

Orient the original polygon so that l is horizontal and $P_2, \ldots, P_{1012}$ and Q lie below l. Note that 1012 edges of the polygon lie on each side of l.



Each of the rays $P_{1013}Q, P_{1014}Q, \ldots, P_{2024}Q, P_1Q$ intersects l. Therefore they intersect the polygon for a second time below l. So the 1013 points $S_{1013}, S_{1014}, \ldots, S_{2024}, S_1$ all lie below l. Hence at most 1011 of the S_i's lie above l. But 1012 edges of the polygon lie above l. So at least one of these edges does not contain an S_i, as required. □

Problem 3. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients such that the leading coefficient of $P(x)$ is 1. Suppose that $P(n)^n$ divides $Q(n)^{n+1}$ for infinitely many positive integers n.

Prove that $P(n)$ divides $Q(n)$ for infinitely many positive integers n.

Solution. Let $R(n) = Q(n)/P(n)$. We have $(R(n))^n Q(n) \in \mathbb{Z}$ for infinitely many integers n. If we suppose that $R(n) \notin \mathbb{Z}$, then $R(n) \in \mathbb{Q}\backslash\mathbb{Z}$, so $R(n) = \frac{q}{p}$ for relatively prime integers q and p, with $p \geq 2$. But then $Q(n)\frac{q^n}{p^n} \in \mathbb{Z}$, which implies that p^n divides $Q(n)$, and in particular $Q(n) \geq p^n \geq 2^n$. But

Q is a polynomial, so there are at most finitely many n for which $Q(n) \geq 2^n$. The result follows. $\square$

Problem 4. Let $ABCD$ be a convex quadrilateral. Prove that there exists a point P inside the quadrilateral such that

$$\begin{aligned} &\angle PAB + \angle PDC = \angle PBC + \angle PAD = \\ &\angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ \end{aligned} \tag{1}$$

if and only if the diagonals AC and BD are perpendicular.

Solution. For a point P distinct from A, B, C and D, let circles (APD) and (BPC) intersect again at Q ($Q \equiv P$ if the circles are tangent). Next, let circles (AQB) and (CQD) intersect again at R. We show that if P lies in $ABCD$ and satisfies (1) then AC and BD intersect at R and are perpendicular; the converse is also true. It is convenient to use directed angles. Let $\angle(UV, XY)$ denote the angle of counterclockwise rotation that makes line UV parallel to line XY . Recall that four noncollinear points U, V, X and Y are concyclic if and only if $\angle(UX, VX) = \angle(UY, VY)$.

The definitions of points P, Q and R imply

$$\begin{aligned} &\angle(AR, BR) = \angle(AQ, BQ) = \angle(AQ, PQ) + \angle(PQ, BQ) = \\ &\angle(AD, PD) + \angle(PC, BC), \\ &\angle(CR, DR) = \angle(CQ, DQ) = \angle(CQ, PQ) + \angle(PQ, DQ) = \\ &\angle(CB, PB) + \angle(PA, DA), \\ &\angle(BR, CR) = \angle(BR, RQ) + \angle(RQ, CR) = \angle(BA, AQ) + \angle(DQ, CD) = \\ &\angle(BA, AP) + \angle(AP, AQ) + \angle(DQ, DP) + \angle(DP, CD) = \\ &\angle(BA, AP) + \angle(DP, CD). \end{aligned}$$

Observe that the whole construction is reversible. One may start with point R, define Q as the second intersection of circles (ARB) and (CRD), and then define P as the second intersection of circles (AQD) and (BQC). The equalities above will still hold true.

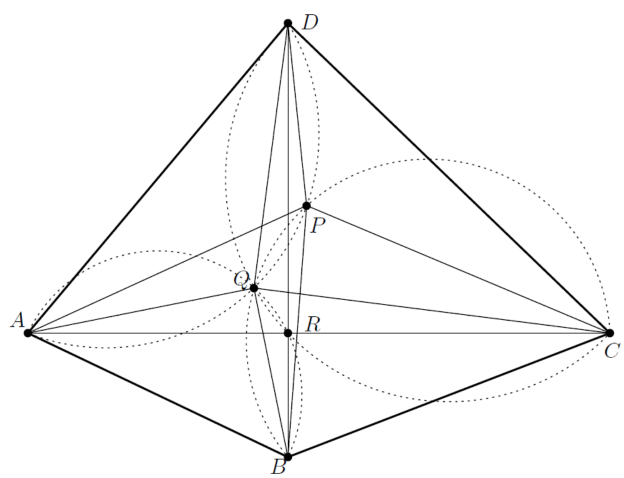
Assume in addition that P is interior to $ABCD$. Then

$$\angle(AD, PD) = \angle PDA,$$
$$\angle(PC, BC) = \angle PCB,$$
$$\angle(CB, PB) = \angle PBC,$$
$$\angle(PA, DA) = \angle PAD,$$
$$\angle(BA, AP) = \angle PAB,$$
$$\angle(DP, CD) = \angle PDC.$$

Suppose that P lies in $ABCD$ and satisfies (1). Then $\angle(AR, BR) = \angle PDA + \angle PCB = 90^\circ$ and similarly $\angle(BR, CR) = \angle(CR, DR) = 90^\circ$. It follows that R is the common point of lines AC and BD, and that these lines are perpendicular.

For the reverse implication suppose that AC and BD are perpendicular and intersect at R. We show that the point P defined by the reverse construction (starting with R and ending with P) lies in $ABCD$. This is enough to finish the solution, because then the angle equations above will imply (1).

One can assume that Q, the second common point of circles (ABR) and (CDR), lies in $\angle ARD$. Then in fact Q lies in $\triangle ADR$ as $\angle AQR$ and $\angle DQR$ are obtuse. Hence $\angle AQD$ is obtuse, too, so that B and C are outside circle (ADQ) ($\angle ABD$ and $\angle ACD$ are acute).



Now $\angle CAB + \angle CDB = \angle BQR + \angle CQR = \angle CQB$ implies $\angle CAB < \angle CQB$ and $\angle CDB < \angle CQB$. Hence A and D are outside circle (BCQ).

In conclusion, the second common point P of circles (ADQ) and (BCQ) lies on their arcs ADQ and BCQ.

We can assume that P lies in $\angle CQD$. Since

$$\angle QPC + \angle QPD = (180^\circ - \angle QBC) + (180^\circ - \angle QAD = \\ 360^\circ - (\angle RBC + \angle QBR) - (\angle RAD - \angle QAR) = \\ 360^\circ - \angle RBC - \angle RAD.$$

point P lies in $\triangle CDQ$, and hence in $ABCD$. The proof is complete. $\square$