# Preparation for EMC 2023

Second Training Test for Senior Category

Solutions

**Problem 1.** Let $I$ be the incenter of $\triangle ABC$, and $P$ be the orthogonal projection of $B$ onto the line $AI$. Let $X$ and $Y$, respectively, be the points of contact of the incircle of $\triangle ABC$ with the sides $BC$ and $AC$. Prove that $P, X, Y$ are collinear.

**Solution.** Let $\alpha, \beta$ and $\gamma$ denote the radian measures of the interior angles $CAB, ABC$ and $BCA$, respectively. Denote by $Q$ the intersection point of lines $AI$ and $XY$. Obviously $Q$ lies on the extension of segment $AI$ on the side of $I$, whereas, in regard to line $XY$, $Q$ belongs to the ray $YX$; moreover, it is readily seen that $Q$ is an interior point of segment $XY$ if and only if $\beta > \gamma$.

We show $Q \equiv P$ by proving that $\angle AQB = \frac{\pi}{2}$. Observe that the latter equality is equivalent to $Q \in (IXZ)$, where $Z$ denotes the point of contact of the incircle of $\triangle ABC$ with side $AB$ and $(IXZ)$ is the circumcircle of $\triangle IXZ$. (Because segment $BI$ is the diameter of $(IXZ)$.)

As tangent segments, $|CX| = |CY|$, which implies $\angle CXY = \frac{\pi}{2} - \frac{\gamma}{2}$. Assume first that $Q$ lies on the extension of segment $YX$ on the side of $X$. Let $R$ be the intersection point $AQ \cap BC$. Then $\angle ARC = \pi - \frac{\alpha}{2} - \gamma$ and $\angle RXQ = \angle CXY$, which in turn imply that $\angle RQX = \frac{\beta}{2} = \angle IZX$. Consequently, $Q \in (IXZ)$.

Assume now that $Q$ is interior point of segment $XY$. Similarly to above, $\angle BXZ = \frac{\pi}{2} - \frac{\beta}{2}$. Thus $\angle YXZ = \frac{\beta}{2} + \frac{\gamma}{2}$. As $\angle ZAQ = \frac{\alpha}{2}$, the sum $\angle AQX + \angle AZX$ of interior angles at $Q$ and $Z$ in quadrilateral $AQXZ$ equals $\frac{3\pi}{2}$. Consequently, because $\angle AZI = \frac{\pi}{2}$, the sum $\angle IQX + \angle IZX$ (of two opposite interior angles in convex quadrilateral $IQXZ$) equals $\pi$. However, this is equivalent to the desired $Q \in (IXZ)$ (since $Z$ and $Q$ lie on opposite sides in regard to line $IX$). $\square$

**Remark.** The content of this problem is usually referred to as *Right angles on incircle chord Lemma.* It's origin is unclear, but it has been a common knowledge among geometers of the 19th century.

**Problem 2.** Cvetko and Spiro play the following game: starting with the number 2 written on a blackboard, each player on turn changes the current number $n$ to a number $n + p$, where $p$ is a prime divisor of $n$. Cvetko goes first and the players alternate on turn. The game is lost by the one who is forced to write a number greater than $\underbrace{2\ldots2}_{2023}$.

Assuming perfect play, who will win the game? (Prove your answer.)

**Solution.** We prove that Cvetko wins the game. For argument's sake, suppose that Spiro can win by proper play regardless of what Cvetko does on each of his moves. Note that Cvetko can force the line $2 \to \mathbf{4} \to 6 \to \mathbf{8} \to 10 \to \mathbf{12}$ at the beginning stages of the game. (As each intermediate 'position' from which Spiro has to play is a prime power.) Thus, if the number 12 is written on the blackboard, then the player on turn must be in a 'winning position', i.e., he can win the game with skillful play. However, Cvetko can place himself in that position through the following line which is once again forced for Spiro: $2 \to \mathbf{4} \to 6 \to \mathbf{9} \to 12$. (This time Cvetko is on turn with 12 written on the blackboard.) The obtained contradiction proves our point. □

**Remark.** The following reasoning validates that the concepts of 'winning position' and 'loosing position' are well defined. In a nutshell, a simple backtracking works. Namely, call every integer greater than $\underbrace{2\ldots2}_{2023}$ a 'winning position' (for the player on turn). As for a number $n \leq \underbrace{2\ldots2}_{2023}$, it is said to be a 'loosing position' (for the player on move) if every number $n + p$, where $p$ is a prime divisor of $n$, represents a 'winning position'; otherwise, $n$ is said to be a 'winning position' itself. Thus, to summarize the solution, regardless of whether 12 is winning or loosing (deciding on this is most likely an arduous task even for a computer), the number 2 is winning because the player on move (Cvetko) can force a line with a prescribed player on turn and the number 12 being written on the blackboard.

**Problem 3.** The real numbers $x_1, x_2, \ldots, x_n$ belong to the interval $[-1, 1]$, and the sum of their cubes is equal to zero. Prove that the sum $x_1 + x_2 + \cdots + x_n$ does not exceed $n/3$.

**Solution.** Note that the inequality $4x^3 - 3x + 1 = (x+1)(2x-1)^2 \geq 0$ holds true for any $x \geq -1$. Therefore, we have the inequality

$$\sum_{k=1}^{n}(4x_k^3 - 3x_k + 1) = -3\sum_{k=1}^{n} x_k + n \geq 0\,,$$

which implies the desired inequality $\sum\limits_{k=1}^{n} x_k \leq n/3$. □

**Problem 4.** Let $a, b, n$ be positive integers, $b > 1$ and $b^n - 1 \mid a$. Show that the representation of the number $a$ in the base $b$ contains at least $n$ digits different from zero.

**Solution.** Let $s$ be the minimum number of nonzero digits that can appear in the $b$-adic representation of any number divisible by $b^n - 1$. Among all numbers divisible by $b^n - 1$ and having $s$ nonzero digits in base $b$, we choose the number $A$ with the minimum sum of digits. Let $A = a_1 b^{n_1} + \cdots + a_s b^{n_s}$, where $0 < a_i \leq b - 1$ and $n_1 > n_2 > \cdots > n_s$. First, suppose that $n_i \equiv n_j \pmod{n}$, for a pair of distinct indices $i, j$. Consider the number

$$B = A - a_i b^{n_i} - a_j b^{n_j} + (a_i + a_j) b^{n_j + kn}\,,$$

with $k$ chosen large enough so that $n_j + kn > n_1$: this number is divisible by $b^n - 1$ as well. But if $a_i + a_j < b$, then B has $s - 1$ digits in base $b$, which is impossible; on the other hand, $a_i + a_j \geq b$ is also impossible, for otherwise $B$ would have sum of digits less for $b - 1$ than that of $A$ (because $B$ would have digits 1 and $a_i + a_j - b$ in the positions $n_j + kn + 1$, $n_j + kn$). Therefore $n_i \not\equiv n_j$ if $i \neq j$. Let $n_i \equiv r_i$, where $r_i \in \{0, 1, \ldots, n-1\}$ are distinct. The number $C = a_1 b^{r_1} + \cdots + a_s b^{r_s}$ also has $s$ digits and is divisible by $b^n - 1$. But since $C < b^n$, the only possibility is $C = b^n - 1$ which has exactly $n$ digits in base $b$. It follows that $s = n$. □