Preparation for EMC 2023

Fifth Training Test for Senior Category

10th December 2023

Problem 1. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the inequality

$$f(n+1) > f(f(n))$$

for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Problem 2. There are 2^n words of length n over the alphabet $\{0, 1\}$. Prove that the following algorithm generates the sequence $w_0, w_1, \ldots, w_{2^n-1}$ of all these words such that any two consecutive words differ in exactly one digit.

(1) $w_0 = 00\ldots0$ (n zeros).

(2) Suppose $w_{m-1} = a_1 a_2 \ldots a_n$, $a_i \in \{0, 1\}$. Let $e(m)$ be the exponent of 2 in the representation of m as a product of primes, and let $j = 1 + e(m)$. Replace the digit a_j in the word w_{m-1} by $1 - a_j$. The obtained word is w_m.

Problem 3. Let P be a point inside $\triangle ABC$ such that

$$\angle APB - \angle C = \angle APC - \angle B\,.$$

Let D, E be the incenters of $\triangle APB$, $\triangle APC$ respectively. Show that AP, BD, and CE meet in a point.

Problem 4. Show that the number of $0-1$ strings of length n with exactly m zeros that are followed immediately by ones equals

$$\binom{n+1}{2m+1}.$$

Allotted time: 4 hours.