Preparation for EMC 2023

Fourth Training Test for Senior Category

3rd December 2023

Problem 1. Ninety-one white pawns are placed on a 10×10 chessboard. Misha repaints these pawns black one at a time and puts down each repainted pawn on an empty square of the board. Prove that eventually two pawns of different colors will occupy two squares that have a common side.

Problem 2. Let A, B, and C be noncollinear points. Prove that there is a unique point X in the plane of ABC such that

$$XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2 .$$

Problem 3. Let a and b be nonnegative integers such that $ab \geq c^2$, where c is an integer. Prove that there exists a natural number n and integers $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ such that

$$\sum_{i=1}^{n} x_i^2 = a, \quad \sum_{i=1}^{n} y_i^2 = b, \quad \text{and} \quad \sum_{i=1}^{n} x_i y_i = c .$$

Problem 4. Let $S(x)$ be the sum of digits of positive integer x in its decimal representation. Find the smallest value of $S(1998n)$ for positive integer n.

Allotted time: 4 hours.