

Preparation for EMC 2023

Third Training Test for Senior Category

30th November 2023

Problem 1. Let a, b, c and d be positive real numbers with $a+b+c+d = 4$. Prove that

$$\frac{ab}{a^2 - \frac{4}{3}a + \frac{4}{3}} + \frac{bc}{b^2 - \frac{4}{3}b + \frac{4}{3}} + \frac{cd}{c^2 - \frac{4}{3}c + \frac{4}{3}} + \frac{da}{d^2 - \frac{4}{3}d + \frac{4}{3}} \leq 4.$$

Problem 2. Let Q be a point inside the convex polygon $P_1P_2 \cdots P_{2024}$. For each $i = 1, 2, \dots, 2024$, extend the line P_iQ until it meets the polygon again at a point S_i . Suppose that none of the points $S_1, S_2, \dots, S_{2024}$ is a vertex of the polygon.

Prove that there is at least one side of the polygon that does not contain any of the points $S_1, S_2, \dots, S_{2024}$.

Problem 3. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients such that the leading coefficient of $P(x)$ is 1. Suppose that $P(n)^n$ divides $Q(n)^{n+1}$ for infinitely many positive integers n .

Prove that $P(n)$ divides $Q(n)$ for infinitely many positive integers n .

Problem 4. Let $ABCD$ be a convex quadrilateral. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ$$

if and only if the diagonals AC and BD are perpendicular.

Allotted time: 4 hours.