

Preparation for EMC 2023

Second Training Test for Senior Category

26th November 2023

Problem 1. Let I be the incenter of $\triangle ABC$, and P be the orthogonal projection of B onto the line AI . Let X and Y , respectively, be the points of contact of the incircle of $\triangle ABC$ with the sides BC and AC . Prove that P, X, Y are collinear.

Problem 2. Cvetko and Spiro play the following game: starting with the number 2 written on a blackboard, each player on turn changes the current number n to a number $n + p$, where p is a prime divisor of n . Cvetko goes first and the players alternate on turn. The game is lost by the one who is forced to write a number greater than $\underbrace{2 \dots 2}_{2023}$.

Assuming perfect play, who will win the game? (Prove your answer.)

Problem 3. The real numbers x_1, x_2, \dots, x_n belong to the interval $[-1, 1]$, and the sum of their cubes is equal to zero. Prove that the sum $x_1 + x_2 + \dots + x_n$ does not exceed $n/3$.

Problem 4. Let a, b, n be positive integers, $b > 1$ and $b^n - 1 \mid a$. Show that the representation of the number a in the base b contains at least n digits different from zero.

Allotted time: 4 hours.