Preparation for EMC 2023

Fifth Training Test for Junior Category

10th December 2023

Problem 1. Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$(1+x)(1+y)(1+z) \geq 2\left(1 + \sqrt[3]{\frac{y}{x}} + \sqrt[3]{\frac{z}{y}} + \sqrt[3]{\frac{x}{z}}\right).$$

Problem 2. Let a, b, c be positive integers such that $a^3 + b^3 = 2^c$. Prove that $a = b$.

Problem 3. Each point in the plane is assigned one of four colours. Prove that there exist two points at distance 1 or $\sqrt{3}$ from each other that are assigned the same colour.

Problem 4. The bisector of angle A of triangle ABC ($AB > AC$) meets its circumcircle at point P. The perpendicular to AC from C meets the bisector of angle A at point K. A circle with center P and radius PK meets the minor arc PA of the circumcircle at point D. Prove that the quadrilateral $ABDC$ has an incircle.

Allotted time: 4 hours.