Preparation for EMC 2023

First Training Test for Junior Category

Solutions

Problem 1. Let a and b be positive numbers. Prove that

$$\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)\geq a\sqrt{b}+b\sqrt{a}.$$

Solution. From AM$-$GM inequality it follows that $\frac{a+b}{2}\geq 2\sqrt{ab}$. Therefore

$$\begin{aligned}
&\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-a\sqrt{b}-b\sqrt{a}\\
&=\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)\\
&\geq\frac{1}{2}(a+b)^2+\frac{1}{4}(a+b)-\frac{a+b}{2}\left(\sqrt{a}+\sqrt{b}\right)\\
&=\frac{a+b}{2}\left[a+b+\frac{1}{2}-\sqrt{a}-\sqrt{b}\right]\\
&=\frac{a+b}{2}\left[\left(\sqrt{a}-\frac{1}{2}\right)^2+\left(\sqrt{b}-\frac{1}{2}\right)^2\right]\geq 0.
\end{aligned}$$

□

Problem 2. Find all positive integers n such that 2^n+7^n is a perfect square.

Solution. Since $2^1+7^1=9=3^2$, $n=1$ is a solution. We show it is the only solution.
For $n>1$, we have

$$2^n\equiv 0 \pmod{4}.$$

We also have

$$7^n\equiv(-1)^n \pmod{4}.$$

Since all perfect squares are either congruent to 0 or 1 modulo 4, the sum $2^n + 7^n$ cannot be a perfect square if n is odd and greater than 1. So write $n = 2m$, where m is a positive integer. We would like to show that $2^n + 7^n$ cannot be a perfect square. Considering this expression modulo 5, we have

$$2^n + 7^n = 4^m + 49^m \equiv 2 \cdot (-1)^m \pmod 5 .$$

Therefore, $2^n + 7^n$ is congruent to 2 or 3 modulo 5. On the other hand, all perfect squares are congruent to 0, 1 or 4 modulo 5. Therefore, $n = 1$ is indeed the only solution to the problem. □

Problem 3. Amy and Bec play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move.
Prove that Bec can always win the game, no matter how Amy plays.

Solution. Call a pile *perilous* if the number of stones in it is one more than a multiple of three, and safe otherwise. Ben has a winning strategy by ensuring that he only leaves Amy perilous piles. Ben wins because the number of stones is strictly decreasing, and eventually Amy will be left with two or three piles each with just one stone.
To see that this is a winning strategy, we prove that Ben can always leave Amy with only perilous piles, and that under such circumstances, Amy must always leave Ben with at least one safe pile.
On Amy's turn, whenever all piles are perilous it is impossible to choose one such perilous pile and divide it into two or three perilous piles by virtue of the fact that

$$1 + 1 \not\equiv 1 \pmod 3 \quad \text{and} \quad 1 + 1 + 1 \not\equiv 1 \pmod 3 .$$

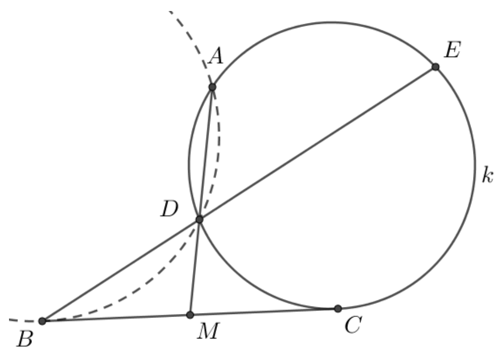
Thus Amy must leave Ben with at least one safe pile.
On Ben's turn, whenever one of the piles is safe, he can divide it into two or three piles, each of which are safe, by virtue of the fact that

$$2 \equiv 1 + 1 \pmod 3 \quad \text{and} \quad 0 \equiv 1 + 1 + 1 \pmod 3 .$$

□

Problem 4. Let ABC be a triangle with $\angle BAC < 90°$. Let k be the circle through A that is tangent to BC at C. Let M be the midpoint of BC, and let AM intersect k a second time at D. Finally, let BD (extended) intersect k a second time at E. Prove that $\angle BAC = \angle CAE$.

Solution.



Since MC is tangent to circle k at C, then by the power of a point theorem we have

$$MC^2 = MD \cdot MA.$$

Since $MB = MC$ it follows that

$$MB^2 = MD \cdot MA.$$

Hence considering the power of M with respect to circle ADB, it follows that MB is tangent to circle ADB at$M$.

In the angle chase that follows, AST is an abbreviation for the alternate segment theorem.

$$\begin{aligned}\angle BAC &= \angle BAM + \angle MAC \\ &= \angle MBD + \angle DAC \quad (AST \quad \text{circle} \quad ADB) \\ &= \angle CBD + \angle BCD \quad (AST \quad \text{circle} \quad k) \\ &= \angle CDE \quad (\text{exterior angle} \quad \triangle BCD) \\ &= \angle CAE \quad (AECD \quad \text{cyclic})\,,\end{aligned}$$

which is the desired result. □