Preparation for EMC 2023

Fifth Training Test for Junior Category

Solutions

Problem 1. Let x, y and z be positive real numbers such that $xyz = 1$. Prove that

$$(1+x)(1+y)(1+z) \geq 2(1 + \sqrt[3]{\frac{y}{x}} + \sqrt[3]{\frac{z}{y}} + \sqrt[3]{\frac{x}{z}}).$$

Solution. Put $a = bx$, $b = cy$ and $c = az$. The given inequality then takes the form

$$(1+\frac{a}{b})(1+\frac{b}{c})(1+\frac{c}{a}) \geq 2(1 + \sqrt[3]{\frac{b^2}{ac}} + \sqrt[3]{\frac{c^2}{ba}} + \sqrt[3]{\frac{a^2}{cb}}) = 2(1 + \frac{a+b+c}{3\sqrt[3]{abc}}).$$

By the AM-GM inequality we have

$$(1+\frac{a}{b})(1+\frac{b}{c})(1+\frac{c}{a}) = \frac{a+b+c}{a} + \frac{a+b+c}{b} + \frac{a+b+c}{c} - 1$$

$$\geq 3(\frac{a+b+c}{\sqrt[3]{abc}}) - 1 \geq 2 \cdot \frac{a+b+c}{\sqrt[3]{abc}} + 3 - 1 = 2(1 + \frac{a+b+c}{\sqrt[3]{abc}}).$$

□

Problem 2. Let a, b, c be positive integers such that $a^3 + b^3 = 2^c$. Prove that $a = b$.

Solution 1. Note that a and b must have the same parity. If a and b are even and $a^3 + b^3$ is a power of two, then $(\frac{a}{2})^3 + (\frac{b}{2})^3$ is also a power of two. But since $\frac{a}{2}$ and $\frac{b}{2}$ are positive integers, $(\frac{a}{2})^3 + (\frac{b}{2})^3$ is of the form 2^d, where d is a positive integer. So if there are distinct positive integers whose cubes sum to a power of two, then one can repeatedly divide them by two to obtain distinct positive odd integers whose cubes sum to a power of two.

So suppose now that a and b are odd. Rewrite the equation as

$$(a+b)(a^2-ab+b^2)=2^c,$$

which implies that there are non-negative integers m and n such that

$$a+b=2^m$$

$$a^2-ab+b^2=2^n.$$

Since a^2-ab+b^2 is odd, we must have $n=0$ and it follows that

$$a+b=2^c=a^3+b^3.$$

However, $a+b \leq a^3+b^3$ with equality if and only if $a=b=1$. Therefore, the only solution to $a^3+b^3=2^c$ with a and b odd is $(a,b,c)=(1,1,1)$. It follows that the only solutions to $a^3+b^3=2^c$ must have $a=b$. □

Solution 2. Let n be the greatest non-negative integer such that $2^n|a$ and $2^n|b$. Write $a=2^nA$ and $b=2^nB$ for positive integers A and B. Then we have

$$2^{3n}(A^3+B^3)=2^c,$$

where at least one of A and B is odd. Since $2^{3n}|2^c$, we have $c=3n+d$ for some non-negative integer d, so $A^3+B^3=2^d$. Since $A,B \geq 1$, we have $d \geq 1$, so $A+B$ is even. Since at least one of A and B is odd, we conclude that both are odd.

So we have

$$2^d=(A+B)(A^2-AB+B^2).$$

Since $2^d, A+B>0$, then we also have

$$A^2-AB+B^2>0.$$

But A^2-AB+B^2 is odd and a factor of 2^d, so

$$A^2-AB+B^2=1.$$

If $A>B$, then

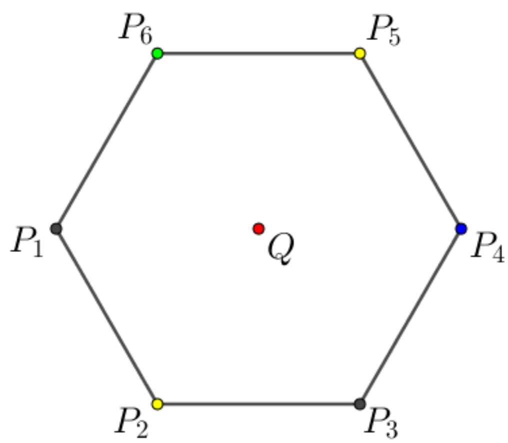
$$A^2-AB+B^2=A(A-B)+B^2 \geq A+B^2 \geq 2,$$

so this case does not occur. Similarly, $A<B$ does not occur.

If $A=B$, it follows that $A=B=1$, and so $a=b$. □

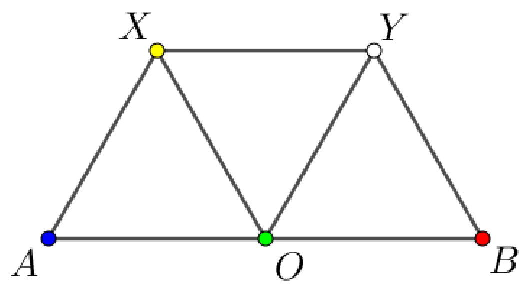
Problem 3. Each point in the plane is assigned one of four colours. Prove that there exist two points at distance 1 or $\sqrt{3}$ from each other that are assigned the same colour.

Solution. Suppose that the four colours are blue, red, yellow, and green. We argue by contradiction and suppose that there do not exist two points at distance 1 or $\sqrt{3}$ from each other that have the same colour.
Pick a point P_1 in the plane and suppose that it is coloured blue, without loss of generality. Construct a regular hexagon $P_1P_2P_3P_4P_5P_6$ with side length 1 and centre Q. Note that the points P_1, P_2, P_6, Q must be coloured differently. So suppose without loss of generality that Q is coloured red, P_2 is coloured yellow, and P_6 is coloured green.



Now note that P_1, P_6, P_5, Q must be coloured differently, which forces P_5 to be yellow. Similarly, P_6, P_5, P_4, Q must be coloured differently, which forces P_4 to be blue. It follows that any point at distance 2 from P_1 must be coloured blue. In other words, there is a circle of radius 2 that is coloured blue. However, there exists a chord on this circle of length 1, which forces two points at distance 1 that are the same colour. This contradicts our original assumption, so it follows that there exist two points at distance 1 or $\sqrt{3}$ from each other that are the same colour. □

Solution 2. Suppose that the four colours are blue, red, yellow, and green. We argue by contradiction and suppose that there do not exist two points at distance 1 or $\sqrt{3}$ from each other that have the same colour.
Consider an isosceles triangle ABC with $BC = 1$ and $AB = AC = 2$. Since B and C must be different colours, one of them is coloured differently to A. Without loss of generality, A is blue and B is red. Orient the plane so that AB is a horizontal segment.



Let O be the midpoint of AB. Then as $AO = BO = 1$, O is not blue or red. Without loss of generality, O is green. Let X be the point above line AB so that $\triangle AOX$ is equilateral. It is easy to compute that $XB = \sqrt{3}$ and $XA = XO = 1$. Hence, X is not red, blue or green, and must be yellow. Finally, let Y be the point above line AB so that $\triangle BOY$ is equilateral. Then it is easy to compute that $YX = YO = YB = 1$ and $YA = \sqrt{3}$. Hence Y cannot be any of the four colours, giving the desired contradiction. $\square$

Problem 4. The bisector of angle A of triangle ABC ($AB > AC$) meets its circumcircle at point P. The perpendicular to AC from C meets the bisector of angle A at point K. A circle with center P and radius PK meets the minor arc PA of the circumcircle at point D. Prove that the quadrilateral $ABDC$ has an incircle.

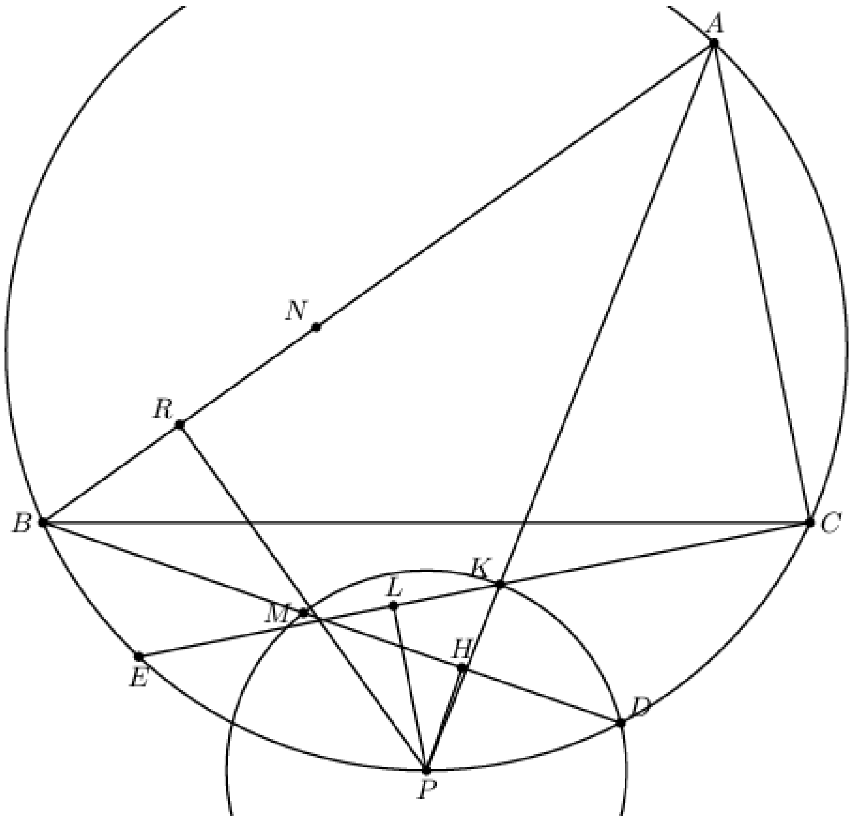
Solution. The arc AB not containing C is greater than the arc AC not containing B because $AB > AC$. The minor arc BP equals the minor arc CP because AP is a bisector. Therefore the arc ACP is less than $180°$ and K lies on the chord EC where E is opposite to A. From this we obtain that $PK < PC$ and so D lie on the minor arc PC.
We have to prove that

$$AB + DC = AC + BD.$$

Let the circle centeredat P with radius PD meet for the second time DB at point M. Construct the perpendicular PH from P to BD. By the Archimedes lemma H bisects the length of the broken line BDC, i.e.

$$BH = HD + DC,$$

but since $MH = HD$ we obtain that $BM = DC$.
Consider the reflection N of C about the bisector AP. We have $AC = AN$, also $PC = PN$, i.e C, N, B lie on a circlecentered at P. Now we have to prove that $BN = DM$.



Let PL be the perpendicular to EC and PR be the perpendicular to BA. Then

$$BN = 2BR = 2PL = 2DH = DM.$$

The second equality is correct because the triangles PRB and CLP are congruent ($\angle BPR = 90^\circ - \angle PBA = \angle PCE = \angle PCL$). The third equality is correct because the triangles PLK and DHP are congruent ($\angle LPK = \angle PAC = \angle PAB = \angle PDH$). $\square$