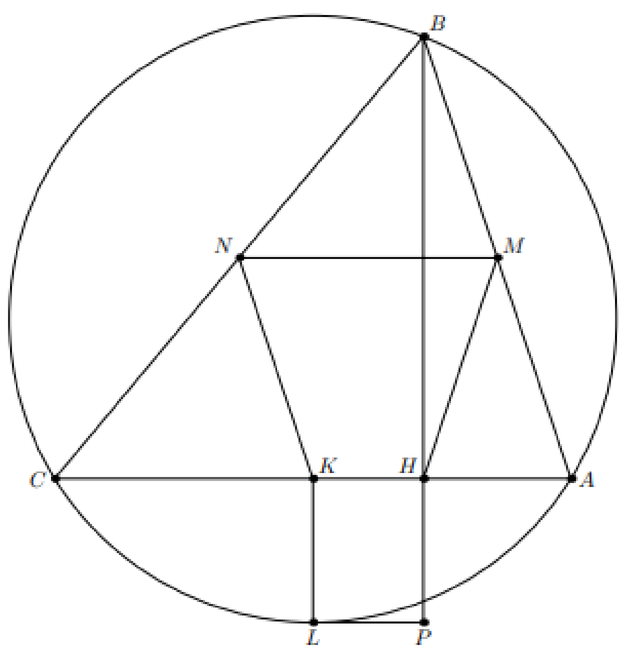
Preparation for EMC 2023

Fourth Training Test for Junior Category

Solutions

Problem 1. Let L be the midpoint of the minor arc AC of the circumcircle of an acute-angled triangle ABC. A point P is the projection of B to the tangent at L to the circumcircle. Prove that P, L, and the midpoints of sides AB, BC are concyclic.

Solution.



Let M, N, and K be the midpoints of AB, BC, and AC, respectively. Let H be the foot of the altitude from B. Then H is also the common point of BP and AC. It is clear that

$$MN \parallel AC \parallel PL,$$

thus $MNLP$ is a trapezoid. It is known that $MNKH$ is an isosceles trapezoid, hence

$$\angle PHM = \angle NKL, \quad MH = KN.$$

Also $PH = KL$. Therefore the triangles MHP and NKL are congruent, i.e the trapezoid $MNLP$ is isosceles, thus P, L, and the midpoints of AB and BC are concyclic.

□

Problem 2. Let $b \geq 2$ be an integer, and let $s_b(n)$ denote the sum of the digits of n when it is written in base b. Show that there are infinitely many positive integers that cannot be represented in the form $n + s_b(n)$, where n is a positive integer.

Solution. Define $S(n) = n + s_b(n)$, and call a number unrepresentable if it cannot equal $S(n)$ for a positive integer n. We claim that in the interval $(b^p, b^{p+1}]$ there exists an unrepresentable number, for any positive integer p.

If b^{p+1} is unrepresentable, then we're done. Otherwise, it is time for our lemma:

Lemma. Define the function $f(p)$ to equal the number of integers x less than b^p such that $S(x) \geq b^p$. If $b^{p+1} = S(y)$ for some y, then $f(p+1) > f(p)$.

Proof. Let $F(p)$ be the set of integers x less than b^p such that $S(x) \geq b^p$. Then for every integer in $F(p)$, append the digit $(b-1)$ to the front of it to create a valid integer in $F(p+1)$. Also, notice that $(b-1) \cdot b^p \leq y < b^{p+1}$. Removing the digit $(b-1)$ from the front of y creates a number that is not in $F(p)$. Hence, $F(p) \to F(p+1)$, but there exists an element of $F(p+1)$ not corresponding with $F(p)$, so $f(p+1) > f(p)$.

Note that our lemma combined with the Pigeonhole Principle essentially proves the claim. Therefore, because there are infinitely many intervals containing an unrepresentable number, there are infinitely many unrepresentable numbers. □

Problem 3. Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

Solution. We first consider the case where one of p, q is even. If $p = 2$, $p - q = 0$ and $pq - q = 2$ which doesn't satisfy the problem restraints. If $q = 2$, we can set $p - 2 = x^2$ and $2p - 2 = y^2$ giving us

$$p = y^2 - x^2 = (y + x)(y - x).$$

This forces $y - x = 1$ so

$$p = 2x + 1 \implies 2x + 1 = x^2 + 2 \implies x = 1$$

giving us the solution $(p, q) = (3, 2)$.

Now assume that p, q are both odd primes. Set $p - q = x^2$ and $pq - q = y^2$ so

$$(pq - q) - (p - q) = y^2 - x^2 \implies p(q - 1) = (y + x)(y - x)$$

Since $y + x > y - x$, $p|(x + y)$. Note that $q - 1$ is an even integer and since $y + x$ and $y - x$ have the same parity, they both must be even. Therefore, $x + y = pk$ for some positive even integer k. On the other hand,

$$p > p - q = x^2 \implies p > x \quad \text{and} \quad p^2 - p > pq - q = y^2 \implies p > y.$$

Therefore, $2p > x + y$ so $x + y = p$, giving us a contradiction.

Therefore, the only solution to this problem is $(p, q) = (3, 2)$.

□

Problem 4. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a\sqrt{c^2+1}} + \frac{1}{b\sqrt{a^2+1}} + \frac{1}{c\sqrt{b^2+1}} > 2.$$

Solution. Denote $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$. Then

$$\frac{1}{a\sqrt{c^2+1}} = \frac{1}{\frac{x}{y}\sqrt{\frac{z^2}{x^2}+1}} = \frac{y}{\sqrt{z^2+x^2}} \geq \frac{2y^2}{x^2+y^2+z^2}$$

where the last inequality follows from the AM-GM inequality

$$y\sqrt{x^2+z^2} \leq \frac{y^2 + (x^2+z^2)}{2}.$$

If we do the same estimation also for the two other terms of the original inequality then we get

$$\frac{1}{a\sqrt{c^2+1}} + \frac{1}{b\sqrt{a^2+1}} + \frac{1}{c\sqrt{b^2+1}} \geq \frac{2y^2}{x^2+y^2+z^2} + \frac{2z^2}{x^2+y^2+z^2} + \frac{2x^2}{x^2+y^2+z^2} = 2.$$

Equality holds only if $y^2 = x^2 + z^2$, $z^2 = x^2 + y^2$ and $x^2 = y^2 + z^2$ what is impossible. □