Preparation for EMC 2023

Third Training Test for Junior Category

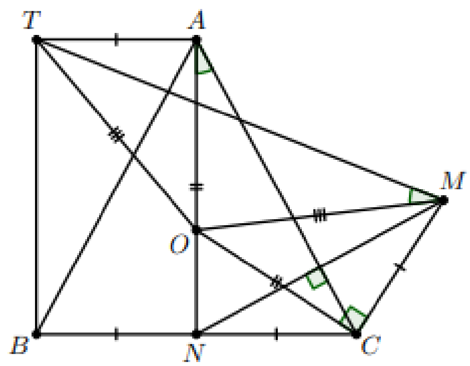
Solutions

Problem 1. A soccer tournament has 2020 teams. Each pair of teams have played each other exactly once. Suppose that no game have led to a draw. The participating teams are ranked first by their points, 3 points for a win and 0 point for a loss; then by their goal difference which is the number of goals scored minus the number of goals against. Is it possible for the goal difference in such ranking to be strictly increasing from top to bottom?

Solution. Assume that such configuration is possible. No two teams can win the Assume that such configuration is possible. No two teams can win the same number of matches, because then the ordering of their goal differences would not satisfy the condition. Each team wins at least 0 and at most 2019 matches. So, the team that finishes k^{th} place wins exactly $2020 - k$ matches. Thus 2020^{th} team always lose, and its goal difference is negative, which implies the goal difference of every team is negative. But it is not possible since the total sum of goal differences equals zero □

Problem 2. Let ABC be an isosceles triangle ($AB = AC$) with its circumcenter O. Point N is the midpoint of the segment BC and point M is the reflection of the point N with respect to the side AC. Suppose that T is a point so that $BNAT$ is a rectangle. Prove that $\angle TMO = \frac{1}{2}\angle BAC$.

Solution.



Since $\triangle ABC$ is an isosceles triangle, we have $\angle CNA = 90^\circ$. Therefore,

$$\angle MCO = \angle MCA + \angle ACO = \angle ACN + \angle ACO = 90^\circ = \angle TAO.$$

Also we have $CM = CN = BN = AT$ and $OC = OA$, so triangles $\triangle OCM$ and $\triangle OAT$ are congruent. Which leads to $OT = OM$ and

$$\angle AOT = \angle COM \implies \angle MOT = \angle COA.$$

Thus, $\triangle AOC \sim \triangle MOT$ and $\angle OMT = \angle OAC = \frac{1}{2}\angle BAC$. $\square$

Problem 3. Prove that the following inequality holds for all positive real numbers x, y and z:

$$\frac{x^3}{y^2+z^2} + \frac{y^3}{z^2+x^2} + \frac{z^3}{x^2+y^2} \geq \frac{x+y+z}{2}.$$

Solution. The inequality is symmetric, so we may assume $x \leq y \leq z$. Then we have

$$x^3 \leq y^3 \leq z^3 \quad \text{and} \quad \frac{1}{y^2+z^2} \leq \frac{1}{z^2+x^2} \leq \frac{1}{x^2+y^2}.$$

Therefore, by the rearrangement inequality we have:

$$\frac{x^3}{y^2+z^2} + \frac{y^3}{x^2+z^2} + \frac{z^3}{x^2+y^2} \geq \frac{y^3}{y^2+z^2} + \frac{z^3}{x^2+z^2} + \frac{x^3}{x^2+y^2}$$

$$\frac{x^3}{y^2+z^2} + \frac{y^3}{x^2+z^2} + \frac{z^3}{x^2+y^2} \geq \frac{z^3}{y^2+z^2} + \frac{x^3}{x^2+z^2} + \frac{y^3}{x^2+y^2}$$

$$\frac{x^3}{y^2+z^2}+\frac{y^3}{x^2+z^2}+\frac{z^3}{x^2+y^2}\geq\frac{1}{2}(\frac{y^3+z^3}{y^2+z^2}+\frac{x^3+z^3}{x^2+z^2}+\frac{x^3+y^3}{x^2+y^2}).$$

What's more, by the rearrangement inequality we have:

$$x^3+y^3\geq xy^2+x^2y$$

$$2x^3+2y^3\geq(x^2+y^2)(x+y)$$

$$\frac{x^3+y^3}{x^2+y^2}\geq\frac{x+y}{2}.$$

Applying it to the previous inequality we obtain:

$$\frac{x^3}{y^2+z^2}+\frac{y^3}{z^2+x^2}+\frac{z^3}{x^2+y^2}\geq\frac{1}{2}(\frac{y+z}{2}+\frac{x+z}{2}+\frac{x+y}{2}).$$

Which is the thesis.

□

Problem 4. For distinct positive integers a, $b<2012$, define $f(a,b)$ to be the number of integers k with $1\leq k<2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a,b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

Solution 1. First we'll show that $S\geq 502$, then we'll find an example (a,b) that have $f(a,b)=502$.

Let x_k be the remainder when ak is divided by 2012, and let y_k be defined similarly for bk. First, we know that, if $x_k>y_k>0$, then $x_{2012-k}\equiv a(2012-k)\equiv 2012-ak\equiv 2012-x_k \pmod{2012}$ and $y_{2012-k}\equiv 2012-y_k \pmod{2012}$. This implies that, since $2012-x_k\neq 0$ and $2012-y_k\neq 0$, $x_{2012-k}<y_{2012-k}$. Similarly, if $0<x_k<y_k$ then $x_{2012-k}>y_{2012-k}$, establishing a one-to-one correspondence between the number of k such that $x_k<y_k$. Thus, if n is the number of k such that $x_k\neq y_k$ and $y_k\neq 0\neq x_k$, then $S\geq\frac{1}{2}n$. Now I'll show that $n\geq 1004$.

If $gcd(k,2012)=1$, then I'll show you that $x_k\neq y_k$. This is actually pretty clear; assume that's not true and set up a congruence relation:

$$ak\equiv bk \pmod{2012}$$

Since k is relatively prime to 2012, it is invertible mod 2012, so we must have $a\equiv b \pmod{2012}$. Since $0<a,b<2012$, this means $a=b$, which the problem doesn't allow, thus contradiction, and $x_k\neq y_k$. Additionally, if $gcd(k, 2012) = 1$, then $x_k \neq 0 \neq y_k$, then based on what we know about n from the previous paragraph, n is at least as large as the number of k relatively prime to 2012. Thus, $n \geq \phi(2012) = \phi(503 * 4) = 1004$. Thus, $S \geq 502$.

To show 502 works, consider $(a, b) = (1006, 2)$. For all even k we have $x_k = 0$, so it doesn't count towards $f(1006, 2)$. Additionally, if $k = 503, 503 * 3$ then $x_k = y_k = 1006$, so the only number that count towards $f(1006, 2)$ are the odd numbers not divisible by 503. There are 1004 such numbers. However, for all such odd k not divisible by 503 (so numbers relatively prime to 2012), we have $x_k \neq 0 \neq y_k$ and $2012 - k$ is also relatively prime to 2012. Since under those conditions exactly one of $x_k > y_k$ and $x_{2012-k} > y_{2012-k}$ is true, we have at most 1/2 of the 1004 possible k actually count to $f(1006, 2)$, so $\frac{1004}{2} = 502 \geq f(1006, 2) \geq S \geq 502$, so $S = 502$.

Solution 2. Let $ak \equiv r_a \pmod{2012}$ and $bk \equiv r_b \pmod{2012}$. Notice that this means $a(2012-k) \equiv 2012 - r_a \pmod{2012}$ and $b(2012-k) \equiv 2012 - r_b \pmod{2012}$. Thus, for every value of k where $r_a > r_b$, there is a value of k where $r_b > r_a$. Therefore, we merely have to calculate $\frac{1}{2}$ times the number of values of k for which $r_a \neq r_b$ and $r_a \neq 0$.

However, the answer is NOT $\frac{1}{2}(2012) = 1006$! This is because we must count the cases where the value of k makes $r_a = r_b$ or where $r_a = 0$.

So, let's start counting.

If k is even, we have either $a \equiv 0 \pmod{1006}$ or $a - b \equiv 0 \pmod{1006}$. So, $a = 1006$ or $a = b + 1006$. We have 1005 even values of k (which is all the possible even values of k, since the two above requirements don't put any bounds on k at all).

If k is odd, if $k = 503$ or $k = 503 \cdot 3$, then $a \equiv 0 \pmod{4}$ or $a \equiv b \pmod{4}$. Otherwise, $ak \equiv 0 \pmod{2012}$ or $ak \equiv bk \pmod{2012}$, which is impossible to satisfy, given the domain $a, b < 2012$. So, we have 2 values of k.

In total, we have $2 + 1005 = 1007$ values of k which makes $r_a = r_b$ or $r_a = 0$, so there are $2011 - 1007 = 1004$ values of k for which $r_a \neq r_b$ and $r_a \neq 0$. Thus, by our reasoning above, our solution is $\frac{1}{2} \cdot 1004 = \boxed{502}$. $\square$