Preparation for EMC 2023

Second Training Test for Junior Category

Solutions

Problem 1. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1\,.$$

Solution 1. Without loss of generality, suppose that $a \leq b$. We have $\frac{a+b}{2} = \sqrt{ab} + 1$, hence $\sqrt{ab}$ is rational. Since it is rational and a, b are integers, it follows that $\sqrt{ab}$ is an integer.

Let $a = m^2 k$ where k is the square-free part of a. Then $b = n^2 k$ for some positive integer $n \geq m$. Substituting these into the original condition yields

$$\frac{m^2k + n^2k}{2} = mnk + 1 \Leftrightarrow (n-m)^2 k = 2.$$

It follows that $n = m + 1$ and $k = 2$. So $(a, b) = (2m^2, 2(m+1)^2)$ or $(a, b) = (2(m+1)^2, 2m^2)$ for some non-negative integer m. This can be easily verified to satisfy the original condition. □

Solution 2. After squaring both sides of

$$(a + b) - 2 = 2\sqrt{ab}$$

and rearranging we get

$$a^2 - (4 + 2b)a + (b - 2)^2 = 0.$$

Solving the quadratic for a gives

$$a = (2 + b) \pm \sqrt{8b}.$$

Since a is an integer, $b = 2n^2$ for some non-negative integer n, and

$$a = 2(1 + n^2 \pm 2n) = 2(n \pm 1)^2.$$

It follows that $(a, b) = (2n^2, 2(n+1)^2)$ or $(a, b) = (2(n+1)^2, 2n^2)$ for some non-negative integer n. This can be easily verified to satisfy the original condition. $\square$

Problem 2. A school has 60 students in year 9 who will be divided into three classes of 20 students. Each student writes a list of three other students that they hope to have in their class.

Can the school always arrange for each student to be in the same class as at least one of the three students on their list?

Solution. The answer is no.

Consider the situation where universally popular students A, B and C are on everyone's list, except that A is not on A's list, B is not on B's list and C is not on C's list. Also suppose that a student D is on A's, B's and C's list. Thus each student now has three students on their lists. We will show that the school cannot fulfill its claim.

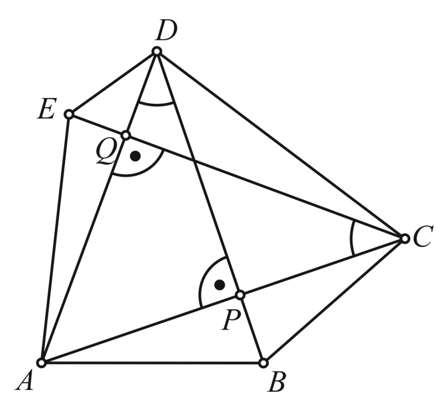
Suppose that one of the classes contains none of A, B, C. Then no one in that class gets any of their preferences.

Otherwise each of the three classes contains one of A, B or C. Without loss of generality we may suppose that D is in the same class as A. Then student B does not get any of their preferences.

Either way the school is unable to fulfill its claim. $\square$

Problem 3. Let $ABCDE$ be a convex pentagon such that AC is perpendicular to BD and AD is perpendicular to CE. Prove that $\angle BAC = \angle DAE$ if and only if $\triangle ABC$ and $\triangle ADE$ have equal areas.

Solution. Let P be the intersection of AC and BD and Q be the intersection of AD and CE.



Since $\angle DPA = \angle AQC = 90^\circ$ we have the similarity $\triangle APD \sim \triangle AQC$, hence $AP : AD = AQ : AC$. Now we have the equivalencies:

$$\begin{aligned}
\angle BAC = \angle DAE & \quad \Leftrightarrow \\
\triangle BAP \sim \triangle EAQ & \quad \Leftrightarrow \\
BP : AP = EQ : AQ & \quad \Leftrightarrow \\
BP : AD = EQ : AC & \quad \Leftrightarrow \\
BP \cdot AC = EQ \cdot AD & \quad \Leftrightarrow \\
P_{\triangle ABC} = P_{\triangle ADE} &
\end{aligned}$$

proving the claim. □

Problem 4. Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}\,, & a_2 &= a_1 + a_3\,, \\
a_3 &= \frac{1}{a_2} + \frac{1}{a_4}\,, & a_4 &= a_3 + a_5\,, \\
&\vdots & &\vdots \\
a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}\,, & a_{2n} &= a_{2n-1} + a_1\,.
\end{aligned}$$

Solution. Notice that by substituting the equations for a_{2k-1} and a_{2k+1} in the equation for a_{2k} we get:

$$a_{2k} = \frac{1}{a_{2k-2}} + \frac{2}{a_{2k}} + \frac{1}{a_{2k+2}} \tag{1}$$

for every index k (here we use $a_{2n+1} = a_1$ and $a_{2n+2} = a_2$). Define m to be the minimum of a_{2k} which we get for a_{2i} and M to be the maximum of a_{2k} which we get for a_{2j}. We have the inequalities:

$$m = a_{2i} = \frac{1}{a_{2i-2}} + \frac{2}{a_{2i}} + \frac{1}{a_{2i+2}} \geq \frac{2}{m} + \frac{2}{M},$$

and

$$M = a_{2j} = \frac{1}{a_{2j-2}} + \frac{2}{a_{2j}} + \frac{1}{a_{2j+2}} \leq \frac{2}{m} + \frac{2}{M}.$$

From these we have $m \geq M$, hence $m = M$ and all a_{2k} are equal. Now (1) implies $a_{2k}^2 = 4$, hence $a_{2k} = 2$ and $a_{2k+1} = \frac{1}{2} + \frac{1}{2} = 1$. We conclude that $(1, 2, 1, \ldots, 1, 2)$ is the only solution. □