

Preparation for EMC 2023

First Training Test for Junior Category

19th November 2023

Problem 1. Let a and b be positive numbers. Prove that

$$\frac{1}{2}(a+b)^2 + \frac{1}{4}(a+b) \geq a\sqrt{b} + b\sqrt{a}.$$

Problem 2. Find all positive integers n such that $2^n + 7^n$ is a perfect square.

Problem 3. Amy and Bec play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move.

Prove that Bec can always win the game, no matter how Amy plays.

Problem 4. Let ABC be a triangle with $\angle BAC < 90^\circ$. Let k be the circle through A that is tangent to BC at C . Let M be the midpoint of BC , and let AM intersect k a second time at D . Finally, let BD (extended) intersect k a second time at E . Prove that $\angle BAC = \angle CAE$.

Allotted time: 4 hours.