

Preparation for EMC 2023

Second Training Test for Junior Category

26th November 2023

Problem 1. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

Problem 2. A school has 60 students in year 9 who will be divided into three classes of 20 students. Each student writes a list of three other students that they hope to have in their class.

Can the school always arrange for each student to be in the same class as at least one of the three students on their list?

Problem 3. Let $ABCDE$ be a convex pentagon such that AC is perpendicular to BD and AD is perpendicular to CE . Prove that $\angle BAC = \angle DAE$ if and only if $\triangle ABC$ and $\triangle ADE$ have equal areas.

Problem 4. Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned} a_1 &= \frac{1}{a_{2n}} + \frac{1}{a_2}, & a_2 &= a_1 + a_3, \\ a_3 &= \frac{1}{a_2} + \frac{1}{a_4}, & a_4 &= a_3 + a_5, \\ &\vdots & &\vdots \\ a_{2n-1} &= \frac{1}{a_{2n-2}} + \frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1} + a_1. \end{aligned}$$

Allotted time: 4 hours.