Preparation for EMC 2023

Third Training Test for Junior Category

30th November 2023

Problem 1. A soccer tournament has 2020 teams. Each pair of teams have played each other exactly once. Suppose that no game have led to a draw. The participating teams are ranked first by their points, 3 points for a win and 0 point for a loss; then by their goal difference which is the number of goals scored minus the number of goals against. Is it possible for the goal difference in such ranking to be strictly increasing from top to bottom?

Problem 2. Let ABC be an isosceles triangle ($AB = AC$) with its circumcenter O. Point N is the midpoint of the segment BC and point M is the reflection of the point N with respect to the side AC. Suppose that T is a point so that $BNAT$ is a rectangle. Prove that $\angle TMO = \frac{1}{2}\angle BAC$.

Problem 3. Prove that the following inequality holds for all positive real numbers x, y and z:

$$\frac{x^3}{y^2+z^2} + \frac{y^3}{z^2+x^2} + \frac{z^3}{x^2+y^2} \geq \frac{x+y+z}{2}.$$

Problem 4. For distinct positive integers a, $b < 2012$, define $f(a,b)$ to be the number of integers k with $1 \leq k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a,b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

Allotted time: 4 hours.