Preparation for EMC 2023

Fourth Training Test for Junior Category

3rd December 2023

Problem 1. Let L be the midpoint of the minor arc AC of the circumcircle of an acute-angled triangle ABC. A point P is the projection of B to the tangent at L to the circumcircle. Prove that P, L, and the midpoints of sides AB, BC are concyclic.

Problem 2. Let $b \geq 2$ be an integer, and let $s_b(n)$ denote the sum of the digits of n when it is written in base b. Show that there are infinitely many positive integers that cannot be represented in the form $n + s_b(n)$, where n is a positive integer.

Problem 3. Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

Problem 4. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a\sqrt{c^2+1}} + \frac{1}{b\sqrt{a^2+1}} + \frac{1}{c\sqrt{b^2+1}} > 2.$$

Allotted time: 4 hours.