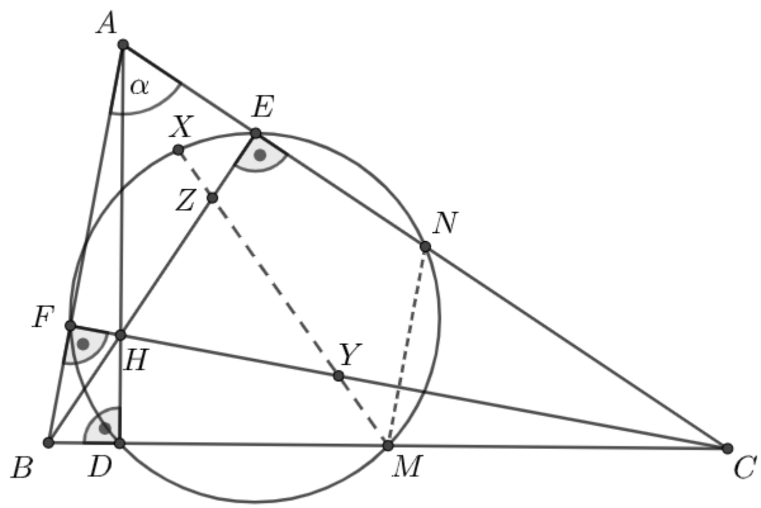
BMO 2021: Macedonian Team Selection Test

Problems and Solutions with marking schemes

1. Let ABC be an acute triangle. Let D, E and F be the feet of the altitudes from A, B and C respectively and let H be the orthocenter of $\triangle ABC$. Let X be an arbitrary point on the circumcircle of $\triangle DEF$ and let the circumcircles of $\triangle EHX$ and $\triangle FHX$ intersect the second time the lines CF and BE second at Y and Z, respectively. Prove that the line YZ passes through the midpoint of BC.

Solution:



Let M and N be the midpoints of the segments BC and AC, respectively. Then $\angle MNA = 180° - \alpha$.

The points M and N are on the nine-point circle for $\triangle DEF$. Thus, the quadrilateral NEFM is cyclic and $\angle MFE = \alpha$. (**2 points**) Since $BCEF$ is a cyclic quadrilateral, the triangle FME is isosceles. (**1 point**) Hence $\angle MEF = \alpha$. From the cyclic quadrilateral $AFHE$ we get $\angle FHE = 180° - \angle MEF$. Then

$$\angle FXZ = 180° - \angle FHZ = 180° - \angle FHE = \angle MEF = \angle MXF.$$

Thus, the points X, Z, and M are collinear. (**3 points**)
Similarly, we show that X, Y and M are collinear. (**3 points**, students should provide the details for this step)

Thus, YZ passes thorough the midpoint of BC. (**1 point**)

2. Define a sequence $x_0 = 1$ and for all $n \geq 0$, $x_{2n+1} = x_n$ and $x_{2n+2} = x_n + x_{n+1}$. Prove that for any relatively prime positive integers a and b, there is a non-negative integer n such that $a = x_n$ and $b = x_{n+1}$.

Solution: Let a and b be relatively prime integers. We will prove the statement by induction on $a+b$. If $a+b=2$, then $a=1$ and $b=1$, so $a=x_0$ and $b=x_1$. (**0 points for the base case.**)

Assume a and b are not both equal to 1. Assume that the statement is true for all relatively prime integers whose sum is less than $a+b$. If $a>b$, then $\gcd(a-b,b)=1$. (**2 points**) By the inductive hypothesis, there is n such that $a - b = x_n$ and $b = x_{n+1}$. Then $a = x_{2n+2}$ and $b = x_{2n+3}$. (**3 points**)

If $a < b$, then $\gcd(a, b-a) = 1$. (**2 points**) By the inductive hypothesis, there is n such that $a = x_n$ and $b - a = x_{n+1}$. Then $a = x_{2n+1}$ and $b = x_{2n+2}$. (**3 points**)

3. Given are non-negative numbers $a_1, a_2, \ldots a_{2021}$ such that $\sum_{k=1}^{2021} a_k = 1$. Prove that

$$\sum_{k=1}^{2021} \sqrt[k]{a_1 a_2 \ldots a_k} \le 3.$$

Solution: Let $b_j = j(1+\frac{1}{j})^j a_j, j = 1, 2, \ldots 2021$. (**5 points**)
Then

$$(k+1)\sqrt[k]{a_1 a_2 \ldots a_k} = \sqrt[k]{b_1 b_2 \ldots b_k} \le \frac{\sum_{j=1}^{k} b_j}{k}. \text{ (1 point)}$$

Therefore,

$$\sum_{k=1}^{2021} \sqrt[k]{a_1 a_2 \ldots a_k} \le \sum_{k=1}^{2021} \frac{1}{k(k+1)} \sum_{j=1}^{k} b_j = \sum_{j=1}^{2021} b_j \sum_{k=j}^{2021} \frac{1}{k(k+1)}. \text{ (1 point)}$$

By simple telescoping we have $\sum_{k=j}^{2021} \frac{1}{k(k+1)} < \frac{1}{j}$. (**1 point**)
Thus,

$$\sum_{k=1}^{2021} \sqrt[k]{a_1 a_2 \ldots a_k} \le \sum_{j=1}^{2021} \frac{b_j}{j} = \sum_{j=1}^{2021} (1+\frac{1}{j})^j a_j < 3 \sum_{j=1}^{2021} a_j = 3. \text{ (2 points)}$$

4. Viktor and Natalia play a colouring game with a 3-dimensional cube taking turns alternatingly. Viktor goes first, and on each of his turns, he selects an unpainted edge, and paints it **v**iolet. On each of Natalia's turns, she selects an unpainted edge, or at most once during the game a face diagonal, and paints it **n**eon green. If the player on turn cannot make a legal move, then the turn switches to the other player. The game ends when nobody can make any more legal moves.

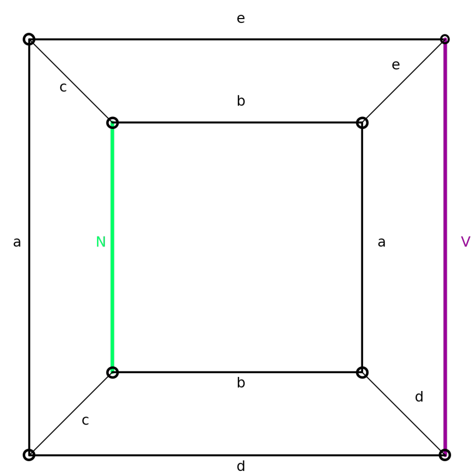
Natalia wins if at the end of the game every vertex of the cube can be reached from every other vertex by traveling only along neon green segments (edges or diagonal), otherwise Viktor wins.

Who has a winning strategy?

Solution: The answer is: Natalia. Consider the cube as a graph Q of order (i.e., number of vertices) 8 and size (i.e., number of edges) 12, and recall the following well known graph-theoretic fact: a graph of order n and size $n-1$ is connected if and only if it is acyclic.

Natalia's strategy can be briefly explained as follows: on each of her first six moves she selects and paints a cube edge while avoiding forming a neon green cycle; then, on her 7th move she adds an appropriate face diagonal to complete a connected acyclic graph (i.e., tree) of order 8. **(1 point)**

Let us fill in the details. On her first move, Natalia paints the edge that is antipodal (symmetric with respect to the center of the cube) to the edge that Viktor painted on his first move; she thus achieves the depicted situation: edges labeled V and N are the ones painted by him and her, respectively, on their first moves. **(2 points)** For the continuation of the game, match up in pairs the remaining ten unpainted edges of the cube as shown below.



On each of her next five moves, Natalia's strategy is to 'complete a pair': she simply paints an edge labeled with the same letter as the one Viktor painted on his previous move. So after six moves from each player, the set of violet edges will have labels V, a, b, c, d, e, and the set of neon green edges will have labels N, a, b, c, d, e. **(4 points)**

To clarify why Natalia surely hasn't completed a neon green cycle, start by observing that no such cycle contains an edge labeled with d or e. Consequently, as Q is a bipartite graph, the only neon green cycle that she could have possibly created must contain precisely 4 edges, labeled with N, a, b, and c in some order. However, each of the two existing 4-cycles in Q that pass through the edge labeled with N contains two edges of the same label (namely, either two b's or two c's).

This proves that Natalia surely avoids forming a neon green cycle during her first six moves. Therefore, the neon green edges of the cube induce a tree of order 7. On her 7th move, Natalia can select appropriately a face diagonal (she has three clear choices) to form a tree of order 8, and thus win. **(3 points)**

Comments:
1) Bulk of the points (4) are awarded for figuring out the pairing of the edges, and 3 more points for proving that this works. 2 points for the first move (the antipodal approach) because this demonstrates going on the right track. 1 point for correct graphical reframing of the problem.
2) There can be an alternative more brute-force approach, and if someone does exhaust all cases, they should get a full score, otherwise at most 2-3 points. A partial solution based on hybrid approach with some brute-forcing and pairing up of vertices can earn up to 4-7 points depending on how close it gets to completion.